Introduction to bosonic string theory

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Abstract

This is an introductory set of five lectures on bosonic string theory. The first one deals with the classical theory of bosonic strings. The second and third lectures cover quantization. Three basic quantization methods are sketched: the old covariant formalism, the light-cone gauge quantization, where the spectrum is derived and the Polyakov path integral formalism. and in particular the partition function at one loop. Finally, the last lecture covers interactions, low energy effective action, the general idea of compactification and in particular toroidal compactification. The notes are based on books by Green, Schwarz and Witten, Polchinski, Lust and Theissen and Kaku and review papers by D'Hocker and Phong and O. Alvarez

1 Classical bosonic string theory

A string is a one dimensional object. It can be closed or open and correspondingly two topologies are possible (see Figure 1).

Figure 1: (a) Closed string; (b) Open string

As it evolves in time it sweeps out a two-dimensional surface in spacetime, which is called the *world sheet* of the string. This is the one dimensional counterpart of the world line for a point particle. Before describing the classical theory of strings it is convenient to first recall the classical mechanics of a zero-dimensional object, a relativistic point particle.

The particle's position can be given in terms of $D-1$ functions of time, $X^i(X^0)$, but this hides the covariance of the theory so it is better to introduce a parameter τ along the particle's world-line and describe the motion in spacetime by D functions $X^\mu(\tau)$. The parametrization is arbitrary: a different parametrization of the same path is physically equivalent, and all physical quantities must be independent of this choice. That is, under $\tau \to \tau'(\tau)$, we must have $X'^\mu(\tau'(\tau)) = X^\mu(\tau)$. A parametrized world line embedded in spacetime is shown in Figure 2.(a).

The simplest Poincaré and reparametrization invariant action would be proportional to the proper time along the world line,

$$S = -m \int d\tau (-\dot{X}^\mu \dot{X}_\mu)^{1/2}, \tag{1}$$

where m is the mass of the particle and a dot denotes a τ derivative. The variation of the action is

$$\delta S = -m \int d\tau \; \dot{u}_\mu \; \delta X^\mu \tag{2}$$

where $u^\mu = \dot{X}^\mu(-\dot{X}^\nu \dot{X}_\nu)^{-1/2}$ is the normalized D-velocity. The equation of motion $\dot{u}^\mu = 0$ thus describes free motion. For simplicity we assume that the spacetime is flat Minkowski space with metric $\eta_{\mu\nu} = \mathrm{diag}(- + \cdots +)$.

Let us introduce canonical conjugate momenta

$$p_\mu = \frac{\delta L}{\delta \dot{X}^\mu} = \frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}. \tag{3}$$

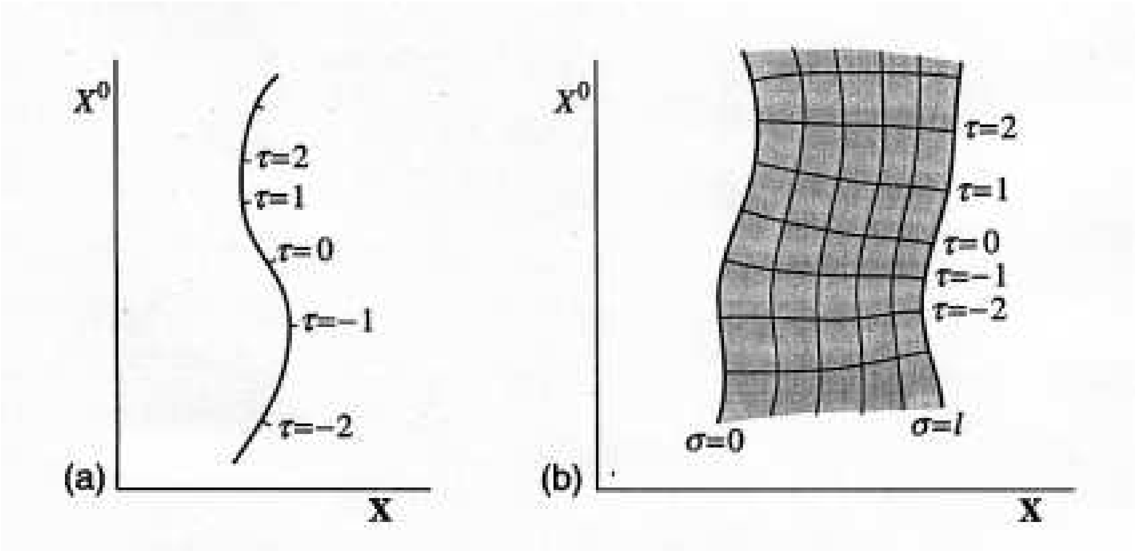

Figure 2: (a) Parametrization of particle world-line. (b) Parametrization of string world-sheet

It is easy to see that these are not all independent. In fact, they satisfy the constraint

$$p^2 + m^2 = 0, \tag{4}$$

which is the mass shell condition. This constraint equation can be obtained as a classical equation of motion from the action

$$S = \int d\tau \left[p_\mu \dot{X}^\mu - \frac{1}{2} e(p^2 + m^2) \right], \tag{5}$$

by varying with respect to e. Quantum mechanically, however, this constraint can be imposed by functionally integrating out over e

$$\int De\ exp\left[-i \int d\tau \frac{1}{2} e(p^2 + m^2)\right] \sim \delta(p^2 + m^2) \tag{6}$$

By functionally integrating (5) over p instead, we obtain another version of the action (1), namely

$$S = \frac{1}{2} \int d\tau\ (e^{-1}\dot{X}^2 - em^2) \tag{7}$$

This action has several advantages over (1). It does not include derivatives inside the square root, it is quadratic in derivatives (and so its path integral can be easily evaluated) and it makes sense for massless particles while (1) does not work in that case.

The classical equivalence between (1) and (7) can be seen by replacing the equation of motion for e,

$$e^2 = -\frac{1}{m^2}\dot{X}^\mu \dot{X}_\mu \tag{8}$$

into the action (7). Quantum mechanically it is easier to work with this latter form of the action.

One can now generalize this analysis of the classical mechanics of the point particle to a one dimensional object. This will sweep out a two dimensional world sheet, which can be described in terms of two parameters, τ and σ, as shown in Figure 2.(b).

The spacetime coordinates of the string are $X^\mu(\tau,\sigma)$. The vectors that are tangent to the world sheet are given by the derivatives of the coordinates, $\partial X^\mu/\partial\tau$; $\partial X^\mu/\partial\sigma$. The contraction of two of these tangent vectors yields a metric, the induced metric on the world sheet

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu \tag{9}$$

where the variables τ, σ have been replaced by $\sigma^{a,b}$, $a, b = 0, 1$. The infinitesimal area on this surface can be written as

$$d\ Area \sim \sqrt{\det|g_{ab}|}\ d\sigma\ d\tau \tag{10}$$

In analogy to the point particle case, where the action is the length swept out by the point, the action of the string now is defined to be the surface area of this world sheet,

$$S_{NG} = \frac{1}{2\pi\alpha'}\int \sqrt{\dot{X}^2 X'^2 - (\dot{X}\cdot X')^2} d\sigma d\tau \tag{11}$$

where the prime represents σ differentiation and the dot τ differentiation. The constant α' is the *Regge slope*. It has units of spacetime length squared and it is related to the tension T of the string by

$$T = \frac{1}{2\pi\alpha'}. \tag{12}$$

The action (11) is called the Nambu-Goto action. Let us consider the symmetries of this action, *i.e.* transformations of $X^\mu(\tau,\sigma)$ which leave the action invariant. These are:

1. D-dimensional Poincaré invariance:

$$X'^\mu(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu, \tag{13}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and a^μ is a translation.

2. Two-dimensional coordinate invariance, often called diffeomorphism ($diff$) invariance. For new world-sheet coordinates $(\tau'(\tau,\sigma), \sigma'(\tau,\sigma))$ the transformation is

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma). \tag{14}$$

The Nambu-Goto action is analogous to the action (1) for the point particle, with derivatives in the square root. Again one can simplify it to get rid of the square root. As before, one can now write down the canonical conjugate momenta of the theory

$$P_\mu = \frac{\delta L}{\delta \dot{X}^\mu} = \frac{1}{2\pi\alpha'} \frac{X'^2 \dot{X}_\mu - (\dot{X}_\nu X'^\nu) X'_\mu}{\sqrt{\det|\partial_a X^\nu \partial_b X_\nu|}}, \tag{15}$$

and it is easy to see that these momenta are not all independent. In fact, they satisfy two identities,

$$P^2 + \frac{1}{(2\pi\alpha')^2} X'^2 = 0 \tag{16}$$

$$P_\mu X'^\mu = 0 \tag{17}$$

Moreover, similarly as in the case of the point particle, the Hamiltonian of the system, $(H = P_\mu \dot{X}^\mu - L)$, vanishes identically. These are indications that we are dealing with a gauge system having an infinite redundancy and that there exists another form of the action expressed in terms of an auxiliary field h_{ab} (the analog of the lagrange multiplier e in the zero dimensional case). This is the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int \sqrt{h}\, h^{ab}\, \partial_a X_\mu\, \partial_b X^\mu\, d\sigma d\tau. \tag{18}$$

It was originally found by Brink, Di Vecchia and Howe [1] and by Deser and Zumino [2], in the course of deriving a generalization with local world-sheet supersymmetry. Its advantages for path integration were emphasized by Polyakov [3]. This is the string counterpart of the point particle action (7) and it is entirely equivalent at the classical level to the earlier Nambu-Goto action. The world sheet metric h_{ab} has Lorentzian signature $(-,+)$. Note that there are two metrics in this form of the action: the two dimensional world sheet metric, $h_{ab}, (a,b = 0,1)$ and the spacetime metric, $\eta_{\mu\nu}, (\mu,\nu = 0,\cdots,D-1)$ (for simplicity we consider string propagation in flat Minkowski spacetime, but later we will see that this can be generalized to an arbitrary metric $G_{\mu\nu}$). Notice also that h_{ab} is not the induced metric g_{ab} defined in equation (9). The classical equivalence between S_{NG} and S_P can be easily worked out.

The action S_P has the following symmetries:

1. D-dimensional Poincaré invariance:

$$X'^{\mu}(\tau,\sigma) = \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + a^\mu, \tag{19}$$

$$h'_{ab}(\tau,\sigma) = h_{ab}(\tau,\sigma). \tag{20}$$

2. Diffeomorphism invariance:

$$X'^{\mu}(\tau',\sigma') = X^{\mu}(\tau,\sigma), \tag{21}$$

$$\frac{\partial\sigma'^{c}}{\partial\sigma^{a}}\frac{\partial\sigma'^{d}}{\partial\sigma^{b}}h'_{cd}(\tau',\sigma') = h_{ab}(\tau,\sigma), \tag{22}$$

for new coordinates $\sigma'^{a}(\tau,\sigma)$.

3. Two-dimensional Weyl invariance:

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma), \tag{23}$$

$$h'_{ab}(\tau,\sigma) = e^{2\rho(\tau,\sigma)}h_{ab}(\tau,\sigma) \tag{24}$$

The variation of the action with respect to the metric defines the energy-momentum tensor,

$$\begin{aligned} T_{ab}(\tau,\sigma) &= -4\pi(-h)^{-1/2}\frac{\delta}{\delta h^{ab}}S_P & (25)\\ &= \frac{1}{\alpha'}(\partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}h_{ab}\partial_c X^{\mu}\partial^c X_{\mu}). & (26)\end{aligned}$$

T_{ab} is conserved, $\nabla^a T_{ab} = 0$, as a consequence of diffeomorphism invariance, and it is traceless, $T^{a}_{\ a} = 0$, as a consequence of Weyl invariance of S_P.

The actions S_{NG} and S_P define two-dimensional field theories on the string world sheet. Most of the machinery we use in string perturbation theory is two-dimensional. From the point of view of the world sheet, the coordinate transformation law (21) defines $X^{\mu}(\tau,\sigma)$ as a scalar field. From the two dimensional point of view S_P describes massless Klein-Gordon scalars X^{μ} covariantly coupled to the metric h_{ab}. Poincaré invariance is an internal symmetry, meaning that it acts on fields at fixed τ,σ. X^{μ} behave as vectors under Poincaré transformations (19).

Varying the action with respect to h_{ab} gives the equation of motion

$$T_{ab} = 0. \tag{27}$$

Note that there are no derivatives of h_{ab} in the action, thus its equation of motion is a constraint.

Varying X^{μ} gives the equation of motion

$$\partial_a(-h)^{1/2}h^{ab}\partial_b X^{\mu} = (-h)^{1/2}\nabla^2 X^{\mu} = 0 \tag{28}$$

For world sheets with boundary there is also a surface term in the variation of the action. Take the coordinate region to be

$$-\infty \le \tau \le \infty, \quad 0 \le \sigma \le l \tag{29}$$

Then,

$$\delta S_P = \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_0^l d\sigma (-h)^{1/2}\delta X_\mu \nabla^2 X^\mu - \frac{1}{2\pi\alpha'}\int_{-\infty}^{\infty} d\tau (-h)^{1/2}\delta X^\mu \partial^\sigma X^\mu|_{\sigma=0}^{\sigma=l}. \quad (30)$$

The boundary term vanishes if

$$\partial^\sigma X^\mu(\tau,0) = \partial^\sigma X^\mu(\tau,l) = 0 \quad (31)$$

These are Neumann boundary conditions on X^μ. Stated more covariantly

$$n^a\partial_a X_\mu = 0 \quad \text{on } \partial M. \quad (32)$$

where n^a is the normal to the boundary ∂M. The ends of the open string move freely in spacetime.

The surface term in the equation of motion will also vanish if we impose

$$X^\mu(\tau,l) = X^\mu(\tau,0) \quad , \qquad \partial^\sigma X^\mu(\tau,l) = \partial^\sigma X^\mu(\tau,0), \quad (33)$$

$$h_{ab}(\tau,l) \;=\; h_{ab}(\tau,0). \quad (34)$$

That is, the fields are periodic. There is no boundary; the endpoints are joined to form a closed loop.

The open string boundary condition (31) and closed string boundary condition (33) are the only possibilities consistent with D-dimensional Poincaré invariance and the equations of motion. If we relax the condition of Poincaré invariance then we allow Dirichlet boundary condition

$$\frac{\partial X^\mu}{\partial\tau} = 0 \quad at\ \sigma = 0 \text{ or } l. \quad (35)$$

This condition can be integrated, and then it specifies a spacetime location on which the string ends. The only way this makes sense is if the open string ends on a physical object: a D-brane (D stands for Dirichlet). If all the open string boundary conditions are Neumann, the ends of the string can be anywhere in spacetime and this means that there are spacetime filling D-branes present.

S_P is not the most general action consistent with all the symmetries of the theory. Demanding that the symmetries be maintained and that the action be polynomial in derivatives can be generalized. Global Weyl invariance, $\rho(\tau,\sigma)$ = constant, requires that the action have one more factor of h^{ab} than h_{ab} to cancel the variation of $(-h)^{1/2}$. The coordinate invariance and Poincaré invariance then allow one additional term beyond the original S_P:

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d\tau d\sigma (-h)^{1/2} R \quad (36)$$

where R is the two dimensional Ricci scalar constructed from h_{ab}. Under a local Weyl rescaling,

$$(-h')^{1/2}R' = (-h)^{1/2}(R - 2\nabla^2\rho). \tag{37}$$

The variation is a total derivative, because $(-h)^{1/2}\nabla_a v^a = \partial_a((-h)^{1/2}v^a)$ for any v^a. The integral (36) is therefore invariant for a world-sheet without boundary. If there are boundaries an additional surface term is needed.

Since χ is allowed by the symmetries it can be included it in the action:

$$S'_P = S_P - \lambda\chi = -\int_{\mathcal{M}} d\tau d\sigma (-h)^{1/2}\left(\frac{1}{4\pi\alpha'}h^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}R\right). \tag{38}$$

This is the most general ($diff\ \times$ Weyl)-invariant and Poincaré invariant action with these fields and symmetries. For the moment the discussion of the symmetries is classical, we are ignoring possible quantum anomalies, which will be considered later.

The action S'_P looks like the Hilbert action for the metric, $\int(-h)^{1/2}R$, minimally coupled to D massless scalar fields X^μ. However, in two dimensions, the Hilbert action depends only on the topology of the world-sheet and does not give dynamics to the metric. To see this, recall that its variation is proportional to $R_{ab} - \frac{1}{2}h_{ab}R$. In two dimensions, the symmetries of the curvature tensor imply that $R_{ab} = \frac{1}{2}h_{ab}R$ and so this vanishes: the Hilbert action is invariant under any continuous change in the metric.

The local invariances of S_P allow for a convenient gauge choice for the world-sheet metric h_{ab}, called conformal gauge. Reparametrization invariance can be used to choose coordinates such that locally $h_{ab} = \Omega^2(\tau,\sigma)\eta_{ab}$. One can then use Weyl invariance to set $h_{ab} = \eta_{ab}$. It is easy to see that the conformal gauge is unique to two dimensions. In $d > 2$ dimensions a metric h_{ab}, being symmetric, has $d(d+1)/2$ independent components. Reparametrization invariance allows to fix d of them, leaving $d(d-1)/2$ components. In two dimensions this is enough to go to conformal gauge. We still have an extra local symmetry, namely Weyl transformations, which allows us to eliminate the remaining metric component.

The general solution of the two-dimensional wave equation (28), specified in the conformal gauge ($(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0$), is given by a sum of "right-movers" and "left-movers",

$$X^\mu(\tau,\sigma) = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma) \tag{39}$$

Let us consider the closed string case first. The fields $X^\mu(\tau,\sigma)$ must be real and must satisfy the periodicity condition $X^\mu(\tau,\sigma + l) = X^\mu(\tau,\sigma)$. These two conditions can be solved explicitly in terms of Fourier series:

$$X_R^\mu = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\ exp\left[\frac{-2i\pi n(\tau - \sigma)}{l}\right]$$

$$X_L^\mu = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + i\,\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^\mu\;exp\left[\frac{-2\pi i n(\tau+\sigma)}{l}\right] \tag{40}$$

$\alpha_n^\mu, \tilde{\alpha}_n^\mu$ are Fourier components that we interpret as coordinate oscillators. x^μ and p^μ can be interpreted as the position and momentum of the center of mass of the string. Indeed it is easy to check that

$$\begin{aligned} x^\mu &= \frac{1}{l}\int_0^l d\sigma X^\mu(\tau,\sigma); \\ p^\mu &= \frac{1}{2l\alpha'}\int_0^l d\sigma\partial_\tau X^\mu(\tau,\sigma) \end{aligned} \tag{41}$$

The reality condition implies that x^μ and p^μ be real and

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger \quad ; \quad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^\dagger \tag{42}$$

It is conventional to define $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$.

The Poisson brackets of the $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ can be easily computed from the Poisson brackets of the X^μ and $\dot{X}^\mu$ at equal τ, and they are

$$[p^\mu, x^\nu]_{PB} = \eta^{\mu\nu} \tag{43}$$

$$[\alpha_m^\mu, \alpha_n^\nu]_{PB} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{PB} = im\delta_{m+n}\eta^{\mu\nu} \tag{44}$$

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{PB} = 0 \tag{45}$$

Thus the Fourier modes α_n^μ for $n \neq 0$ are harmonic oscillator coordinates.

In the case of the open string the solution to the wave equation with Neumann boundary conditions is

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu\tau + i\,\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\;exp\left[\frac{-\pi i n\tau}{l}\right]\cos\frac{n\pi\sigma}{l} \tag{46}$$

(This can be obtained from the solution for the closed string by imposing $X'^\mu = 0$ at $\sigma = 0, l$; the left and right components combine into stationary waves). Therefore, unlike the closed string case, there is only one set of modes in the open string. In this case it is conventional to identify the momentum of the center of mass with the zero mode as $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$.

Let us consider the D-dimensional Poincaré invariance. Since the Poincaré transformations are global symmetries from the point of view of the two dimensional theory, they are associated with conserved currents. Using the well known Noether procedure the currents associated with translation and Lorentz invariance can be obtained. They are respectively

$$P_a^\mu = T\partial_a X^\mu \tag{47}$$

and

$$J_a^{\mu\nu} = T(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu). \tag{48}$$

These currents are conserved

$$\partial_a P^{a\mu} = \partial_a J^{a\mu\nu} = 0 \tag{49}$$

and they describe the D-dimensional momentum and angular momentum of the string. The total conserved momentum and angular momentum of a string are found by integrating these currents over σ at $\tau = 0$, for example

$$P^\mu = T \int_0^l d\sigma \frac{dX^\mu(\sigma)}{d\tau} \tag{50}$$

and

$$J^{\mu\nu} = T \int_0^l d\sigma \left(X^\mu \frac{dX^\nu}{d\tau} - \frac{X^\nu dX^\mu}{d\tau} \right) \tag{51}$$

The wave equation has to be supplemented with the constraint (27)

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu = 0. \tag{52}$$

In components,

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \tag{53}$$

$$T_{01} = T_{10} = \dot{X} \cdot X' = 0 \tag{54}$$

Note that the trace vanishes identically, $h^{ab} T_{ab} = 0$.

The Hamiltonian of the theory is

$$H = \int_0^l d\sigma (\dot{X} \cdot P - L) = \frac{1}{4\pi\alpha'} \int_0^l d\sigma (\dot{X}^2 + X'^2) \tag{55}$$

which can be written in terms of oscillators, for the open string, as

$$\begin{aligned} H = \frac{1}{2\pi\alpha'} \sum_n \sum_m \alpha_n \cdot \alpha_m e^{-i(m+n)\tau} \int_0^l d\sigma \, \cos \frac{n\pi\sigma}{l} \cos \frac{m\pi\sigma}{l} \\ - \sum_{n\neq 0} \sum_{m\neq 0} \alpha_n \cdot \alpha_m e^{-i(n+m)\tau} \int_0^l d\sigma \, \sin \frac{n\pi\sigma}{l} \, \sin \frac{m\pi\sigma}{l} \\ = \frac{1}{2} \sum_n \alpha_n \cdot \alpha_{-n} \end{aligned} \tag{56}$$

In the case of the closed string the same procedure leads to

$$H = \frac{1}{2} \sum_n (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \tag{57}$$

The constraint $H = 0$ allows to write the mass of the string in terms of oscillator modes. For the open string we have

$$0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_n \cdot \alpha_{-n} \tag{58}$$

Thus, we obtain the mass shell condition

$$-\alpha' p^2 = \alpha' M^2 = \sum_{n>0} \alpha_n \cdot \alpha_{-n}, \tag{59}$$

and

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \tag{60}$$

for closed strings.

Most string calculations are carried out on a Euclidean world-sheet where the world-sheet metric h_{ab} is replaced with δ_{ab}. Consider the world-sheet of a closed string, the cylinder parametrized by $\sigma \in [0, 2\pi]$ and $\tau \in [-\infty, +\infty]$. Make a Wick rotation $\tau \to -i\tau$ so that $\sigma^{\pm} = \tau \pm \sigma \to i(-i\tau \pm \sigma)$. In this case it is useful to introduce complex coordinates

$$z' = \tau + i\sigma \quad , \quad \bar{z}' = \tau - i\sigma \tag{61}$$

We can now map the cylinder to the complex plane by

$$z = e^{z'} = e^{\tau + i\sigma} \quad , \quad \bar{z} = e^{\bar{z}'} = e^{\tau - i\sigma} \tag{62}$$

(see Figure 3). Having the theory defined on the complex plane we can now apply all the powerful techniques of complex analysis. Lines of equal time τ are mapped into circles around the origin. Integrals over σ will be replaced by contour integrals around the origin. The infinite past becomes $z = 0$ and infinite future $z = \infty$. σ translations become rotations and time translations become dilatations.

The complex indices are raised and lowered with

$$h_{z\bar{z}} = h_{\bar{z}z} = \frac{1}{2}, \quad h_{zz} = h_{\bar{z}\bar{z}} = 0, \quad h^{z\bar{z}} = h^{\bar{z}z} = 2, \quad h^{zz} = h^{\bar{z}\bar{z}} = 0. \tag{63}$$

Note also that

$$d^2z = 2\ d\sigma\ d\tau \tag{64}$$

with the factor 2 from the Jacobian ($d^2z\ |\det h|^{1/2} = d\sigma\ d\tau$).

In this notation the action is

$$S_P = \frac{1}{2\pi\alpha'} \int d^2z\ \partial X^\mu \bar{\partial} X_\mu \tag{65}$$

and the classical equation of motion is

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0. \tag{66}$$

Writing it as

$$\partial(\bar{\partial}X^\mu) = \bar{\partial}(\partial X^\mu) = 0 \tag{67}$$

it follows that ∂X^μ is holomorphic and $\bar{\partial}X^\mu$ is antiholomorphic (holomorphic in $\bar{z}$).

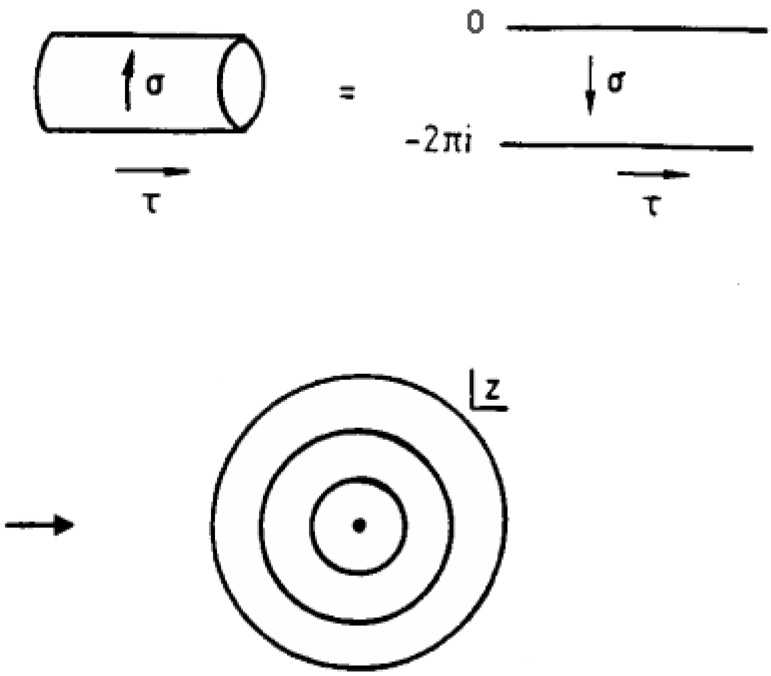

Figure 3: Conformal map

Under the Minkowski continuation $\tau \to -i\tau$, a holomorphic field becomes a function only of $\tau + \sigma$ and an antiholomorphic field a function only of $\tau - \sigma$. We thus use as synonyms

$$\text{holomorphic} = \text{left} - \text{moving}, \quad \text{antiholomorphic} = \text{right} - \text{moving} \tag{68}$$

In complex coordinates, the constraints (52) are $T_{zz} = 0$, $\bar{T}_{\bar{z}\bar{z}} = 0$, and the trace is $T_{z\bar{z}} = 0$. The conservation law $\partial^a T_{ab} = 0$ yields

$$\begin{aligned} \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} &= 0 \\ \partial_z T_{\bar{z}z} + \partial_{\bar{z}} T_{zz} &= 0 \end{aligned} \tag{69}$$

Since $T_{z\bar{z}} = 0$, these equations imply that $T_{zz}(T_{\bar{z}\bar{z}})$ is a holomorphic (antiholomorphic) function. Thus it can be Laurent expanded as

$$T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad ; \quad \bar{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\bar{L}_m}{\bar{z}^{m+2}} \tag{70}$$

The coefficients are called Virasoro generators and can be obtained by inverting these expressions,

$$L_m = \int_{\mathcal{C}} \frac{dz}{2\pi i z} z^{m+2} T_{zz}(z) \tag{71}$$

and similarly for $\bar{L}_m$. $\mathcal{C}$ is any contour enclosing the origin counterclockwise. This is the same as an ordinary Fourier expansion in coordinates σ in $\tau = 0$.

The vanishing trace of T is typical of a theory with conformal invariance. The conservation law (69) corresponds to the existence of an infinite set of conserved quantities. Indeed there is an infinite symmetry generated by currents

$$j(z) = v(z)T(z) \quad ; \quad \bar{j}(\bar{z}) = \bar{v}(\bar{z})\bar{T}(\bar{z}) \tag{72}$$

The conserved quantities correspond to residual symmetries which remain after fixing the conformal gauge. In fact, under diffeomorphisms $\delta h_{ab} = \partial_a \xi_b + \partial_b \xi_a$, and under Weyl transformations $\delta h_{ab} = 2\rho h_{ab}$. Therefore, even after fixing the conformal gauge it is possible to keep the flat metric by combining coordinate and Weyl transformations. This residual symmetry will be used to introduce the light-cone gauge in the next lecture.

Replacing the mode expansions (40) into (71) we obtain

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \quad , \quad \bar{L}_m = \frac{1}{2} \sum_n \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \tag{73}$$

Note that $H = L_0$ for open strings and $H = L_0 + \bar{L}_0$ for closed strings.

The combination $L_0 - \bar{L}_0$ must vanish according to the constraint equations. This implies that the two terms in (60) give equal contributions.

The Poisson brackets of the Virasoro operators can be easily calculated from the Poisson brackets of the oscillators, and we obtain the Virasoro algebra

$$[L_m, L_n]_{PB} = i(m-n)L_{m+n} \tag{74}$$

and similarly for $\bar{L}_m$, while $[L_m, \bar{L}_m] = 0$. This is the algebra satisfied by the generators of the residual symmetries that preserve the conformal gauge condition (infinitesimal diffeomorphisms of S^1). These formulae will be modified later by quantum anomalies.

2 Quantized bosonic string theory

We will discuss the first quantization of the string. This should be contrasted with second quantized string field theory which is based on operators that create or destroy entire strings.

There are several possible formalisms to quantize string theory and it is useful to see the basic features of all of them. There are two covariant quantizations. One is based on a description in terms of the X^μ coordinates, which are considered as quantum mechanical operators. The other one is the modern path integral quantization which has a deeper geometrical basis and is more convenient to compute loop diagrams. The third formalism is the light cone approach, which is not manifestly Lorentz invariant but gives a manifestly unitary theory, free of ghosts.

2.1 Old covariant quantization

Let us start with the more traditional covariant approach. A standard method to go from classical to quantum theory is to replace the Poisson brackets by commutators, substituting

$$[\cdots]_{PB} \to -i\,[\ ,\ ] \tag{75}$$

Thus, from (45) we obtain the commutators

$$[p^\mu, x^\nu] = \eta^{\mu\nu} \tag{76}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde\alpha^\mu_m, \tilde\alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu} \tag{77}$$

$$[\alpha^\mu_m, \tilde\alpha^\nu_n] = 0. \tag{78}$$

The operators α^μ_m, $\tilde\alpha^\mu_m$ can be naturally interpreted as lowering and raising harmonic oscillators for $m > 0$ or $m < 0$, respectively. The ground state $|0; k>$ is defined to be annihilated by the lowering operators and to be an eigenstate of the center of mass momenta,

$$\alpha^\mu_m|0; k>= 0, \quad m > 0 \tag{79}$$

$$p^\mu|0; k>= k^\mu|0; k> . \tag{80}$$

The α^μ_m oscillators are related to a^μ_m the conventional harmonic oscillator operators by $\alpha^\mu_m = \sqrt{m}\ a^\mu_m$, $\alpha^\mu_{-m} = \sqrt{m}\ a^{\mu\dagger}_m$, for $m > 0$. Note that $|0; 0>$ is the ground state of a single string with zero momentum, it is not the vacuum state.

A general state can be built by applying creation operators $\alpha^{\mu\dagger}_m$ to the ground state $|0; k>$. The Fock space defined in this way is not positive definite. The commutation relations of the time components,

$$\left[a^0_m, a^{0\dagger}_m\right] = -1 \tag{81}$$

imply that the state $a_m^{0\dagger}|0;k>$ has negative norm because

$$< 0|a_m^0 a_m^{0\dagger}|0 >= -1 \tag{82}$$

These states are called ghosts and they lead to conflict with the probabilistic interpretation of quantum mechanics.

However, the actual physical space is smaller. We have not yet imposed the constraints which we found in the classical theory, the vanishing of the energy-momentum tensor $T_{ab} = 0$. We have to impose them now as operators, as subsidiary conditions on the states. We then hope that the ghosts decouple from the physical Hilbert space.

Recall that the Fourier modes of T_{ab} are the Virasoro generators

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n \tag{83}$$

Now the α_n^μ are operators so we should resolve the normal ordering ambiguities. Since α_{m-n}^μ commutes with α_n^ν unless $m = 0$, the only ambiguity arises for L_0. We define

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n - A \tag{84}$$

i.e. we have put the lowering operators on the right and the raising operators on the left, and included an unknown constant A from the commutators. In the classical theory, the constraint conditions amount to the vanishing of the Fourier components of T_{ab}, $L_m = \bar{L}_m = 0, \forall m$. Quantum mechanically this can be naturally extended to the condition that L_m annihilate the physical states.

Before doing this however we have to determine the algebra of the L_m. Due to the normal ordering the calculation has to be done with care.

We find (see exercise 3)

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \tag{85}$$

c is called the central charge, and it arises as a quantum effect. Here $c = D$, the dimension of spacetime. This means that each free scalar field contributes one unit to the central charge. Later we will determine the contribution of other fields.

It is now easy to see that even though in the classical theory the constraints are $L_m = 0, \forall m$, this cannot be implemented on the states $|\phi>$ since

$$< \phi|\left[L_m, L_{-m}\right]|\phi >=< \phi|2mL_0|\phi > + \frac{D}{12}m(m^2-1) < \phi|\phi > \tag{86}$$

i.e. we cannot require $L_m|\phi >= 0, \forall m$. The most we can do is to demand that on physical states

$$L_m|phys >= 0, \quad m \geq 0 \tag{87}$$

For the closed string we also have the $\bar{L}_m$'s. They also satisfy a Virasoro algebra and commute with the L_m's. So we impose the conditions (87) for the $\bar{L}_m$'s and in addition

$$(L_0 - \bar{L}_0)|phys>= 0 \tag{88}$$

(this requires $A = \bar{A}$). The operator $U_\epsilon = e^{i\epsilon(L_0 - \bar{L}_0)}$ generates rigid translations in σ (*i.e.* it satisfies $U_\epsilon^\dagger X^\mu(\sigma,\tau)U_\epsilon = X^\mu(\sigma+\epsilon,\tau)$). Since no point on a closed string should be distinct we have to impose (88)

In the classical theory, the condition $L_0 = 0$ gives the mass of the string in terms of the oscillators. Here, for the open string

$$L_0 = \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - A, \tag{89}$$

thus $M^2 = -p^\mu p_\mu$ and the constraint $L_0|phys>= 0$ imply that at the mass of the physical states is given by

$$M^2|phys>= \frac{1}{\alpha'}(N - A)|phys> \tag{90}$$

where we have defined the level number $N = \sum_{m>0} \alpha_{-m} \cdot \alpha_m$.

For the closed string we find from the expressions for L_0 and $\bar{L}_0$

$$M^2 = M_L^2 + M_R^2 = \frac{2}{\alpha'}(\tilde{N} + N - 2A) \tag{91}$$

$$M_L^2 = M_R^2 \tag{92}$$

the second line following from (88).

The mass of the ground state is determined, both in the open and closed string theories, by the normal ordering constants. These constants drop in the expressions for the angular momentum operators and the Poincaré algebra holds at the quantum level. Furthermore, since $[L_m, J^{\mu\nu}] = 0$ the physical state conditions are invariant under Lorentz transformations and the physical states form Lorentz multiplets

One can prove a no ghost theorem which states that the ghosts decouple in 26 dimensions if $A = 1$. In this covariant formalism one can only see indications of extra symmetry for these values. We will arrive at the same conclusion from the light cone quantization of the theory, and later (next lecture) from the path integral formalism.

2.2 Light cone quantization

When we fixed the conformal gauge $h_{ab} = \eta_{ab}$ we saw that there are still residual symmetries left. These can be used to choose other gauges. In the light cone gauge, the Virasoro constraint equations can be solved explicitly and the theory contains only physical states. It is not manifestly covariant but it does not contain ghosts. The spectrum can be shown to be equivalent to the one obtained in the covariant formulation. We will not do this here, but we will discuss the spectrum in the light cone.

Let us introduce light cone coordinates in spacetime

$$X^+ = \frac{X^0 + X^{D-1}}{\sqrt{2}} \quad , \quad X^- = \frac{X^0 - X^{D-1}}{\sqrt{2}} \tag{93}$$

where we have arbitrarily, in a non covariant way, chosen X^{D-1}. The remaining coordinates $X^i, i = 1, \cdots, D-2$, are transverse coordinates.

Light-cone gauge the residual symmetries that are left after fixing the conformal gauge are used to choose $\tau \propto X^+$. Recall that the reparametrizations satisfying $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \propto h_{\alpha\beta}$ can be compensated by a Weyl rescaling. Expressed in light cone coordinates ($\sigma^\pm = \tau \pm \sigma$) the gauge preserving diffeomorphisms satisfy $\partial_+ \xi^- = \partial_- \xi^+ = 0$, *i.e.* $\xi^\pm = \xi^\pm(\sigma^\pm)$. The residual symmetry corresponds to the possibility of reparametrizing $\sigma^+ \to \tilde{\sigma}(\sigma^+)$, $\sigma^- \to \tilde{\sigma}^-(\sigma^-)$, or infinitesimally, $\tilde{\sigma}^\pm = \sigma^\pm + \xi^\pm(\sigma^\pm)$. In terms of $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ and $\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$ and thus,

$$\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) \tag{94}$$

$$\tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+ - \tilde{\sigma}^-) \tag{95}$$

The first equation states that $\tilde{\tau}$ may be an arbitrary solution of the wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)\tilde{\tau} = 0. \tag{96}$$

This is the equation obeyed (in conformal gauge) by the spacetime coordinates $X^\mu(\tau, \sigma)$. Thus, the gauge freedom corresponds to the fact that we can make a reparametrization so that $\tilde{\tau}$ equals one of the X^μ. $\tilde{\tau} = \frac{X^+}{p^+} + const$. This is usually expressed by

$$X^+(\tau, \sigma) = x^+ + p^+\tau. \tag{97}$$

Classically it corresponds to choosing the oscillators $\alpha_n^+ = 0$ for $n \neq 0$.

Now the Virasoro conditions $(\dot{X} \pm X')^2 = 0$ become

$$(\dot{X}^- \pm X'^-) = \frac{1}{2p^+}(\dot{X}^i \pm X'^i)^2 \tag{98}$$

where we used that the nonzero components of the Minkowski metric are $\eta_{+-} = \eta_{-+} = -1$, $\eta_{ij} = \delta_{ij}$, and $\dot{X}^+ = p^+$. This equation can be solved for X^- as a function of

X^i, so that in the light-cone gauge X^+ and X^- can be eliminated, leaving only the transverse oscillators X^i. Using the mode expansion of X^-

$$X^- = x^- + p^-\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma \tag{99}$$

the explicit solution of (98) is

$$\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: - a\delta_n\right) \tag{100}$$

where a normal ordering constant has been introduced for α_0^-. The mass shell condition in light-cone gauge is obtained using the identification of α_0^- with p^-. For $n = 0$

$$p^+p^- = \frac{1}{2}\alpha_0^i\alpha_0^i + \sum_{m=1}^{\infty}\alpha_{-m}^i\alpha_m^i - a \tag{101}$$

Thus

$$M^2 = 2p^+p^- - p_ip_i = 2\sum_{m=1}^{\infty}\alpha_{-m}^i\alpha_m^i - 2A = 2(N - A) \tag{102}$$

where $N = \sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i$ only has contributions from the transverse oscillators.

In the light-cone gauge all the string excitations are generated by the transverse oscillators α_n^i. The first excited state is

$$\alpha_{-1}^i|0;k> \tag{103}$$

which is a $(D-2)$-component vector representation of the transverse rotation group $SO(D-2)$. A transversely polarized vector, subjected to a Lorentz transformation, acquires a longitudinal polarization in general unless it is massless. This is the fact that a massless particle corresponds to an irreducible representation of $SO(D-2)$. Thus, the light-cone gauge cannot give a Lorentz invariant theory unless the vector state (103) is massless, *i.e.* the parameter A must equal 1.

Let us now try to establish the restriction over D. First we consider a heuristic argument that uses the information that A must equal 1 for Lorentz invariance. The normal ordering constant can be computed directly from

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty}\alpha_{-n}^i\alpha_n^i = \sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty} :\alpha_{-n}^i\alpha_n^i: + \frac{D-2}{2}\sum_{n=1}^{\infty} n \tag{104}$$

The infinite sum over n above must be regularized. The 'zeta function regularization' yields $\sum_{n=1}^{\infty} n = -1/12$. Thus

$$A = 1 = \frac{D-2}{24} \tag{105}$$

implies $D=26$. This result, namely that $A=1$, $D=26$ are necessary and sufficient conditions for Lorentz invariance, can be proved more rigorously by a systematic study of the Lorentz generators $J^{\mu\nu}$,

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n) \tag{106}$$

The commutation relations among the $J^{\mu\nu}$ can be explicitly computed in the light cone gauge and an anomaly appears in $[J^{i-}, J^{j-}]$. Lorentz invariance requires that this commutator must vanish, but direct calculation yields

$$\left[J^{i-}, J^{j-}\right] = -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m) \tag{107}$$

with

$$\Delta_m = m\left(\frac{26-D}{12}\right) + \frac{1}{m}\left(\frac{D-26}{12} + 2(1-A)\right) \tag{108}$$

Requiring $\Delta_m = 0$, $\forall m$, gives $D=26$ and $A=1$ as expected.

2.3 Spectrum of the bosonic string theory

The states in the theory are generated by acting with the transverse oscillators on the ground state. We have to distinguish between open and closed strings.

Open string spectrum

The ground state is $|0;k>$. Its mass is given by the operator $\alpha' M^2|0;k> = -A|0;k>$, and thus $A=1$ implies it is a tachyon. The presence of the tachyon in the spectrum means that the vacuum is unstable and the theory might be inconsistent. We will see later that including fermionic degrees of freedom on the world-sheet in a supersymmetric way and requiring spacetime supersymmetry will lead to a tachyon free theory.

The first excited state, $\alpha^i_{-1}|0;k>$, is a $D-2$ dimensional massless vector of the transverse rotation group $SO(D-2)$. Lorentz invariance requires that physical states fall into representations of the little group of the Lorentz group $SO(D-1,1)$ which is $SO(D-1)$ for massive particles and $SO(D-2)$ for massless particles. The light-cone gauge has the disadvantage of giving the states as multiplets of $SO(D-2)$, even though the proof of Lorentz invariance, which holds in $D=26$, guarantees that the massive levels fill out complete multiplets of $SO(D-1)$.

The higher excited states of the string, which are massive, do form full representations of $SO(D-1)$. One can check this by hand for a few levels, and in a more complete analysis it follows from the existence of the full set of Lorentz generators.

The first states with positive mass occur at level 1, and are given by

$$\alpha^i_{-2}|0;k> \quad \text{and} \quad \alpha^i_{-1}\alpha^j_{-1}|0;k>, \tag{109}$$

These are $D-2$ and $(D-2)(D-1)/2$ states, respectively. The sum $(D-2)(D+1)/2$ is the dimensionality of a symmetric traceless representation of $SO(D-1)$, which must be the complete answer therefore.

At level $M^2 = 2/\alpha'$ the possible states are

$$\alpha^i_{-3}|0;k>, \quad \alpha^i_{-2}\alpha^j_{-1}|0;k>, \quad \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0;k> \tag{110}$$

a total of $24 + 576 + 2600 = 3200$ states in all. These combine to give representations of $SO(25)$. Similarly, for $\alpha' M^2 = 3$ one obtains a total of 25650 states in all.

The endpoints of the open string are special points, so it is possible to furnish them with non-dynamical degrees of freedom. For instance one can put 'charges', a 'quark' at one end and an 'antiquark' at the other end, and introduce a $U(n)$ symmetry group under which the 'quark' and 'antiquark' transform in the n and $\overline{n}$ representations respectively. In this case, a basis of string states is $|k;ij>$, where $i,j = 1,...,n$ correspond to the $U(n)$ states of the 'quark' and 'antiquark'. There are now n^2 massless states and the corresponding operators carry $U(n)$ generators λ^a_{ij}, known as the Chan Paton factors.

Closed string spectrum

The spectrum of closed string states is easily deduced from that of the open strings. Closed strings in the light-cone gauge are described by two sets of transverse oscillators, α^i_n and $\tilde{\alpha}^i_n$. In addition one has the restriction that $L_0 = \tilde{L}_0$, the level matching condition. Thus the closed string multiplet with $\alpha' M^2 = 4(N-1)$ is given by tensor products of open string states having $\alpha' M^2 = 2(N-1)$ with themselves.

The ground state is a tachyon with $\alpha' M^2 = -4$. The next level is a set of massless states of the form

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k> \tag{111}$$

with $SO(24)$ quantum numbers corresponding to the tensor product of a massless vector of $SO(24)$ from left moving modes with a massless vector of $SO(24)$ from right moving modes. The part of (111) which is symmetric and traceless in i and j transforms under $SO(24)$ as a massless spin two particle: this is the graviton. The trace term $\delta_{ij}\alpha^i_{-1}\alpha^j_{-1}|0;k>$ is a massless scalar, usually called the dilaton. Finally, the antisymmetric part transforms under $SO(24)$ as an antisymmetric second rank tensor. One can also go on to describe closed string states of positive mass squared by taking suitable products of left-right moving open string states.

The spectrum described above is that of the oriented closed strings. It is also possible to restrict the spectrum to states corresponding to an unoriented string. Physically, an oriented string carries an intrinsic 'arrow' while an unoriented string is directionless. Mathematically, an unoriented string is a string whose wave functional $\Psi(X^\mu(\sigma))$ is invariant under $\sigma \to -\sigma$. The quantum wave functional of an oriented string has no such restrinction, thus so far we have discussed oriented closed string theory. Let us briefly discuss the unoriented case. The concept of orientability can be made precise by defining an unitary operator T which reverses the orientation of a string, *i.e.* $T^\dagger X^\mu(\sigma,\tau)T = X^\mu(l-\sigma,\tau)$. Since $\sigma \to -\sigma$ interchanges left and right movers, the state $|\Psi>$ of an unoriented string must be symmetric under exchange of the two sets of oscillators. For example, at the massless level the antisymmetric tensor term should be dropped, whereas the graviton and dilaton states should be kept. For the open string, the unoriented theory contains states with even mode number only, *i.e.* the odd levels are absent and especially it has no massless states. In terms of oscillators this means

$$T^\dagger \alpha_n^\mu T = (-1)^n \alpha_n^\mu \quad \text{(open string)} \tag{112}$$

$$T^\dagger \alpha_n^\mu T = \tilde{\alpha}_n^\mu \qquad \text{(closed string)} \tag{113}$$

We have seen that it is possible to introduce gauge groups in the open string theory, for example $U(n)$. Classically other groups are possible if one considers unoriented open strings, in particular $SO(n)$ and $Sp(n)$.

All the bosonic string theories are sick as they stant, because the spectrum contains a tachyon. A tachyon means that one is doing perturbation theory about an unstable vacuum. It is conceivable that the closed string tachyon condenses in an analogous manner to the Higgs mechanism: in the case of the unbroken symmetry extremum of the Higgs potential in the standard model there is a stable minimum, where the Higgs fields acquire a vacuum value. Recently there has been suceess in demostrating that open string tachyons condense at a stable minimum, but the fate of closed string tachyons is still an open problem.

In order to interpret this mass spectra we will next consider interactions.

2.4 Path integral quantization

So far we have discussed the spectrum of string theory, but in order to fully describe the theory we must also include the interaction between various particles in the spectrum. In particular we would like to know how to compute a scattering amplitude involving various string states. It turns out that, unlike point particle theories, there is a unique way of introducing interaction in string theory, it results from splitting and joining strings. In the point particle theories, the dynamics of the freely moving particle and of the interaction of several particles are separate components of the theory: in particular, the smooth world lines of free propagation experience a singular joining

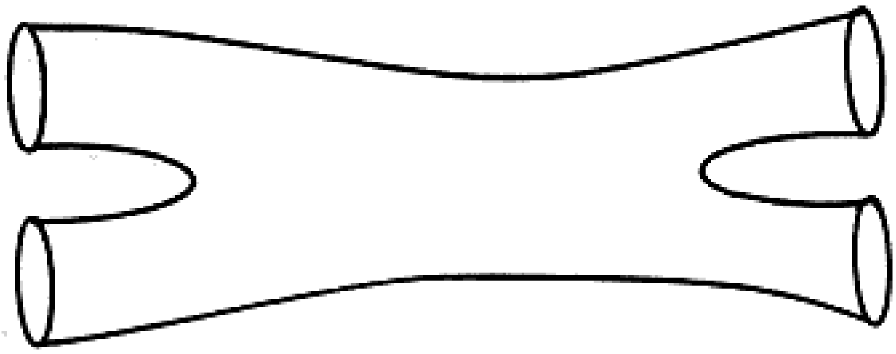

Figure 4: Two strings joining into one string and then splitting into two strings.

at the interaction point. The nature of the interaction is an additional input in the theory. In string theory, instead, the local dynamics of the string does not depend on whether there are interactions or not. The topology of the world-sheet alone informs us that the strings interact.

Consider for example a scattering involving four external strings (see Figure 4). If we imagine the time axis running from left to right in Figure 4, then this diagram represents two strings joining into one string and then splitting into two strings, the analog of a tree diagram in field theory. A more complicated surface is shown in Figure 5. This represents two strings joining into one string, which then splits into two and joins again, and finally splits into two strings. This is the analog of a one loop diagram in field theory. The relative normalization between the contributions from these two diagrams is not determined by any consistency requirement. This introduces an arbitrary parameter in string theory, known as g_s, the string coupling constant. However, once the relative normalization between these two diagrams is fixed, the relative normalization between all other diagrams is fixed due to various consistency requirements. Thus, besides the dimensionful parameter α', string theory has a single dimensionless coupling constant. As we shall see later, both these parameters can be absorbed into definitions of various fields in the theory.

The world-sheets in Figures 4 and 5 have tubes extending into the past and future corresponding to incoming and outgoing strings. Conformal invariance plays a crucial role in the computation of these diagrams because it allows to consider compact world-sheets instead of surfaces with boundaries. The incoming and outgoing strings of Figure 4 can be conformally mapped to points of the two dimensional surface (see Figure 6).

Consider for example the case of a world-sheet with only one incoming and one

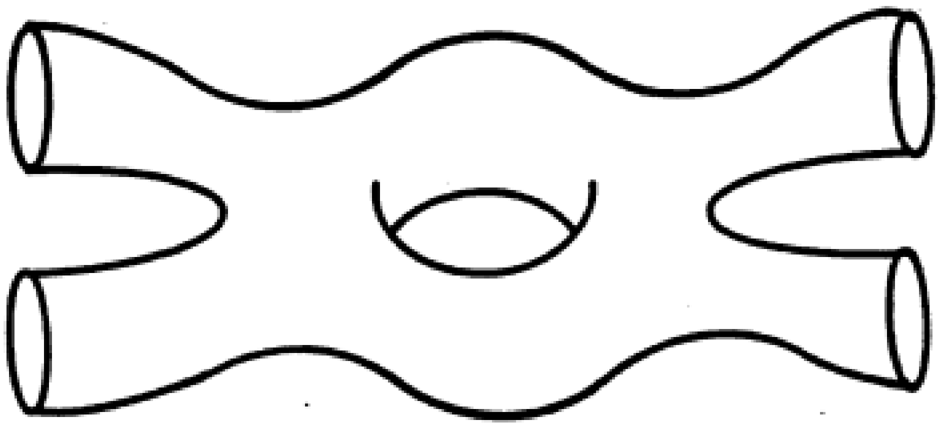

Figure 5: Two strings joining into one string which then splits into two and joins again and finally splits into two strings.

outgoing closed string, (Figure 7.(b)) described by a cylinder with metric $ds^2 = d\tau^2 + d\sigma^2$, $-\infty < \tau < \infty$, $0 \leq \sigma \leq 2\pi$. Taking $\tau = \ln r$ this becomes $ds^2 = r^{-2}(dr^2 + r^2 d\sigma^2)$. The incoming string ($\tau = -\infty$) has been mapped to the point $r = 0$ and the outgoing string ($\tau = \infty$) to $r = \infty$. The string world-sheet has been mapped to the plane. A suitable choice of a conformal factor allows to map the plane to the sphere. We rescale the metric by $4r^2(1+r^2)^{-2}$ and get $ds^2 = 4\frac{dr^2 + r^2 d\sigma^2}{(1+r^2)^2} = \frac{4dzd\bar{z}}{(1+|z|^2)^2}$ where $z = re^{i\sigma}$. This is the standard round metric of the sphere, stereographically projected onto the plane. Indeed with $z = \cot(\theta/2)e^{i\phi}$ we find $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. The incoming and outgoing strings are now finite points, namely the south and north pole of the sphere.

The same idea holds for external open string states. The long strips in Figure 7. (a) represent world sheets of open strings whose endpoints define the boundaries. It can be mapped to the unit disk with the incoming and outgoing strings located on the boundary.

For more complicated string diagrams with several incoming and outgoing strings, the conformal factor can always be chosen to map all of them to points on the sphere. This remains also true for loop diagrams where the external strings are mapped to points on spheres with g handles if g is the number of loops. The quantum numbers of the external string states are generated by local operators inserted at these points, which are called vertex operators. In summary, performing a conformal mapping, the world-sheet becomes a two-dimensional surface with the incoming and outgoing particles inserted by local vertex operators. In this sense vertex operators can be

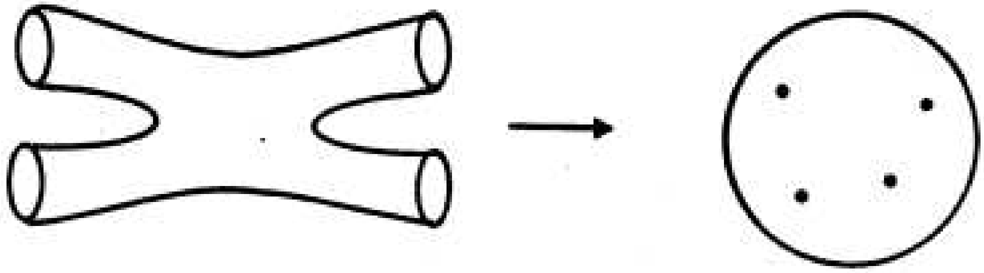

Figure 6: Conformal mapping of incoming and outgoing strings to points on the two dimensional sphere.

viewed as conformal projections of asymptotic states. It is however only known how to construct vertex operators for on-shell states.

Two dimensional surfaces are topologically completely characterized by g, their number of handles, b the number of boundaries and c the number of crosscaps. A crosscap corresponds to cutting a hole in the surface and identifying diametrically opposite points (this operation does not include boundaries in the surface). In terms of these the Euler number is given as

$$\chi = \frac{1}{4\pi}\int \sqrt{h}\; R\; d^2\sigma + \frac{1}{2\pi}\int_{\partial\mathcal{M}} ds\; K = 2(1-g) - b - c. \tag{114}$$

where K is the geodesic curvature. g is also called the *genus* of the surface (*e.g.* $g = b = c = 0$ for the sphere, S^2; $g = c = 0, b = 1$ for the disk, D^2; $g = 1, b = c = 0$ for the torus, T^2; $g = c = 0, b = 2$ for the cylinder, C^2; $g = 0, b = c = 1$ for the Mobius strip, M^2; $g = b = 0, c = 2$ for the Klein bottle, K^2; etc.). In summary, a n-point, g-loop amplitude is described by a two-dimensional sphere with g handles and n vertex operator insertions.

To compute a general n-point amplitude one has to sum in the functional integral over all compact surfaces. There are actually four ways to define the sum over world sheets, which correspond to the four free string theories listed above:

1. Closed oriented: all oriented world-sheets without boundary

2. Closed unoriented: All world-sheets without boundaries

3. Closed plus open oriented: All oriented world-sheets with any number of boundaries.

4. Closed plus open unoriented: All world-sheets with any number of boundaries.

Notice that this list does not include any theory with open strings but not closed

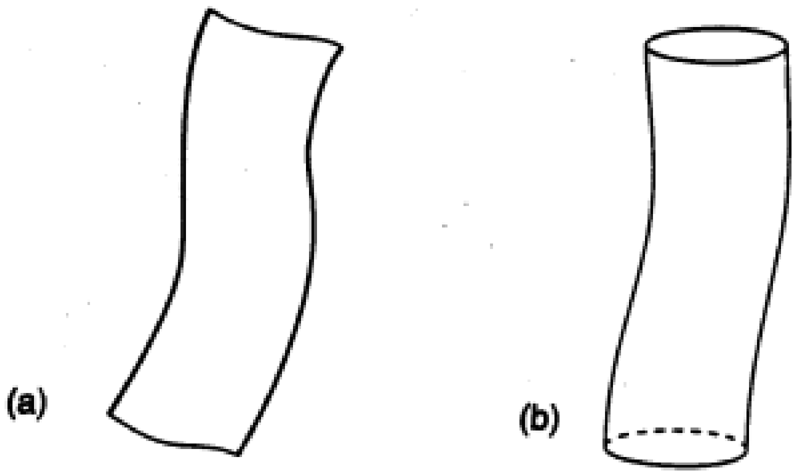

Figure 7: Figure 7. (a) Free propagation of open string; (b) Free propagation of closed string.

strings. Let us see why this is so. Consider a world sheet with the topology of an annulus (Figure 8. (a)). From the dashed line we see that this is a process with an intermediate state of two open strings, and so world-sheets of this topology must be present in any theory with open strings. In Figure 8.(b) the same topology is drawn as a cylinder and cut open on an intermediate state of a single closed string. Thus even if we start with open strings only, the sum over world sheets will necessarily include processes in which scattering of open strings produces closed strings.

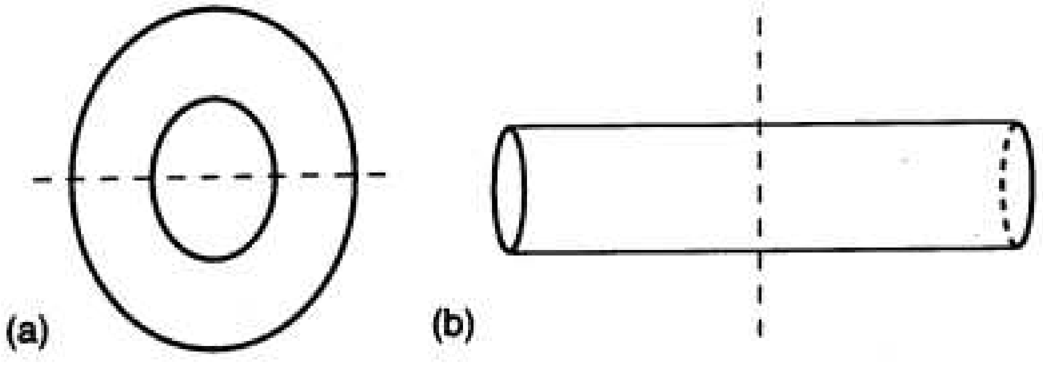

Figure 8: Figure 8. (a) Annulus cut open; (b) Cilynder cut open.

The need for closed strings in open string theory can also be seen in another way. Consider the interactions in figures 9. (a) and (b), but time reversed: respectively two open strings joining into one, and an open to closed string transition. Near the interaction point these are the same, two endpoints coalescing. To have the first interaction without the second would require some nonlocal constraint on the dynamics of the string; this would surely be inconsistent. One can make the same argument with figures 9. (c) and (d), where the interaction is locally the reconnection of a pair of strings. So if any open string interaction is allowed, then so is some process in which closed strings are produced from open strings.

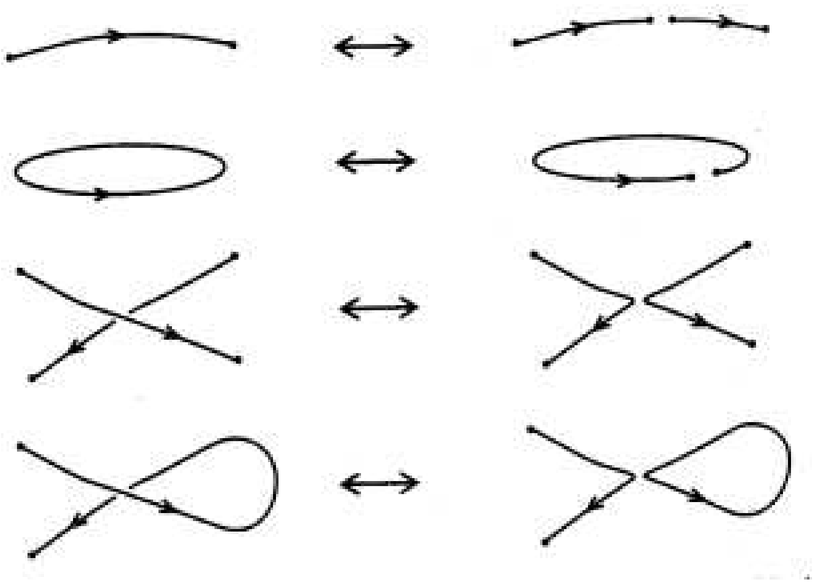

Figure 9: Figure 9. (a) Annulus cut open; (b) Cilynder cut open.

2.4.1 The critical dimension in the path integral formalism

The prescription for computing scattering amplitudes is then to take the weighted sum over all possible string world-sheets bounded by the external strings with weight factor e^{-S_P}. Conformal invariance allows to compute these amplitudes of on-shell particles as correlation functions of vertex operators. In the Polyakov formulation this corresponds to treating both the X^μ and the world-sheet metrics h_{ab} as two dimensional quantum fields, *i.e.* the following path integral has to be evaluated:

$$A_n = \sum_{g=0}^{\infty} A_n^{(g)}$$

$$= \sum_{g=0}^{\infty} \int \mathcal{D}h\mathcal{D}X^\mu \int_M \prod_{i=1}^{N_c} d^2 z_i V_{\alpha_i}(z_i, \bar{z}_i) \prod_{j=1}^{N_o} \int_{\partial M} d^2 z_j V_{\beta_j}(z_j, \bar{z}_j) e^{-S_P[X,h]}$$

$$= \sum_{g=0}^{\infty} e{-}\lambda\chi \int_M \prod_{i=1}^{N_c} d^2 z_i \int_{\partial M} \prod_{j=1}^{N_o} d^2 z_j < \prod_{i=1}^{N_c} V_{\alpha_i}(z_i, \bar{z}_i) \prod_{j=1}^{N_o} V_{\beta_j}(z_j, \bar{z}_j) > \tag{115}$$

where $V_\alpha(z, \bar{z})$ represent vertex operators that describe emission or absorption of a closed string state of type α from $\mathcal{M}$, the interior of the string world sheet, and V_β is a vertex operator that describes emission or absorption of an open string state of type β from $\partial\mathcal{M}$, the boundary of the string world sheet. These vertex operators are tipically of the form

$$V(k^\mu, X^\mu(z)) = P(\varepsilon, DX^\mu(z, \bar{z}), \bar{D}X^\mu(z, \bar{z}))e^{ik\cdot X(z,\bar{z})} \tag{116}$$

where $P(\varepsilon, DX^\mu(z, \bar{z}), \bar{D}X^\mu(z, \bar{z}))$ is a polynomial expression in the derivatives of X and ε is a polarization tensor. This form is dictated by the symmetries of the action, and a key requirement is Weyl invariance after inclusion of anomalies. The lowest mass level vertex operators are

$$\begin{aligned} k^2 = 2(\text{tachyon}), \quad & V(k^\mu) = e^{ik\cdot X} \\ k^2 = 0, \quad & V(k^\mu) = \varepsilon_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu e^{ik\cdot X} \\ k^2 = 0, \quad & V(k^\mu) = \varepsilon_\mu \partial_\tau X^\mu e^{ik\cdot X} \end{aligned} \tag{117}$$

where the operators for massless states in the last two lines correspond to the closed and open strings respectively.

A_0 is called the partition function and it is commonly denoted by Z. Let us consider this object first.

The path integral

$$Z = \int \mathcal{D}X\mathcal{D}h e^{-S_P} \tag{118}$$

has a huge overcounting. Fields (X, h) and (X', h') related by Weyl and reparametrization symmetries represent the same physical configuration. Thus we have to divide by the volume of this local symmetry group, *i.e.* we have to compute

$$Z = \int \frac{\mathcal{D}X\mathcal{D}h}{V_{diff} \times V_{Weyl}} e^{-S_P} \tag{119}$$

The way to do this is by fixing the gauge, integrating over a slice intersecting each equivalence class only once, and this will give us the correct measure over the slice by the standard Fadeev-Popov method.

The integration measures $\mathcal{D}h$ and $\mathcal{D}X$ have to be determined by requirements of symmetry and locality. We will see that it is not possible to respect the symmetries of the classical theory and that the conformal anomalies only cancel in 26 dimensions. In fact the critical dimension will arise in this formalism as the requirement of quantum Weyl invariance. We will see that the scale factor develops an effective dynamics that is described by the Liouville theory, and that it only decouples in $D = 26$ dimensions.

The measures are difened from the norm similarly as in a finite dimensional space the metric $ds^2 = g_{ij}dx^i dx^j$ from which we define the invariant integration measure $d^n x\sqrt{g}$ The measure for $\mathcal{D}X$ is completely determine once one has a metric function on the space of small variations δX^μ, so that one can measure lengths and angles and hence volumes. This metric is unique due to Poincaré and reparametrization invariance,

$$||\delta X^\mu||^2 = \int_{\mathcal{M}} d^2\sigma\sqrt{h}\ \delta X^\mu \delta X_\mu \tag{120}$$

Note that it is not Weyl invariant, and this is the first hint to the Weyl anomaly.

The functional integral in X is easily performed once the measure is specified. It is convenient to divide the field X^μ into a constant zero mode X_0^μ of the Laplacian $\Delta_h = -\frac{1}{\sqrt{h}}\partial_a\sqrt{h}h^{ab}\partial_b$, and all other modes X'^μ orthogonal to it. Thus the Gaussian integral is

$$\int \mathcal{D}Xe^{-S_P} = \int \mathcal{D}X_0 \int \mathcal{D}X' e^{-T\int_{\mathcal{M}} d^2\sigma\sqrt{h}h^{ab}\partial_a X'^\mu \partial_b X'_\mu} = V\left(\frac{\det'\Delta_h}{\int\sqrt{h}d^2\sigma}\right)^{-D/2} \tag{121}$$

where $V = \int \mathcal{D}X_0$ is the volume of spacetime, and the prime in the determinant denotes the exclusion of the zero mode. We have used the identity

$$1 = \int \mathcal{D}Xe^{-||x||^2} = \left[\frac{\pi}{\int d^2\sigma\sqrt{h}}\right]^{D/2} \int \mathcal{D}X' e^{-||X'||^2} \tag{122}$$

If there are vertex operators in the path integral, the spacetime volume element should be replaced by the total momentum conservation δ function, resulting from the X_0^μ integral over the $exp(ik\cdot X)$ factors in the vertex operators.

The integration over metrics is more difficult and we will only sketch the calculation (see references [4, 5] for details). Here also it is necessary to define a precise measure to be used in this integral, and anomalies potentially arise because there is no satisfactory way to define the measure so as to preserve all of the apparent symmetries of S_P. The analysis below applies to closed string theory. For open string theory there are subtleties associated with boundary conditions which we do not discuss.

An infinitessimal deformation δh_{ab} of a metric is a symmetric two-tensor, and the natural norm for δh_{ab} is

$$||\delta h_{ab}||^2 = \int_{\mathcal{M}} d^2\sigma\sqrt{h}(Ch^{ab}h^{cd} + h^{ac}h^{bd})\delta h_{ab}\delta h_{cd} \tag{123}$$

The measure $\mathcal{D}h$ will be associated with this norm. Note that it not Weyl invariant.

We have seen that locally the world-sheet metric can be made Euclidean by combining reparametrizations and Weyl transformations. Thus the three components of

h_{ab} can be trade by the scale factor ρ and the two components of the diffeomorphisms ξ. An infinitesimal deformation δh_{ab} of the metric is given by the joint action of reparametrizations and Weyl transformations as

$$\delta h_{ab} = (2\delta\rho + \nabla_c \xi^c) h_{ab} + (P_1 \xi)_{ab} \tag{124}$$

where the operator P_1 maps vectors into symmetric traceless two-tensors,

$$(P_1 \xi)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a - h_{ab} \nabla_c \xi^c, \tag{125}$$

and describes the traceless piece of the deformation coming from reparametrization by the diffeomorphism ξ^a. The zero modes of this operator are called conformal Killing vectors.

But there can be global obstructions in general. There are metrics which can not be conformally related; *i.e.* they cannot be obtained from each other by reparametrizations and Weyl rescalings. To see this consider the adjoint of P_1, $P_1^\dagger$, which maps traceless symmetric tensors to vectors as

$$(P_1^\dagger t)_a = -2\nabla^b t_{ab} \tag{126}$$

Now, zero modes of $P_1^\dagger$ correspond to symmetric traceless tensors which cannot be written as $(P_1 \xi)_{ab}$ for any vector field ξ. Indeed if $(P^\dagger t_0)_a = 0$, then for all ξ^a, $(\xi, P_1^\dagger t_0) = -(P_1 \xi, t_0) = 0$. This means that the zero modes of $P_1^\dagger$ correspond to metric deformations which cannot be absorbed by reparametrizations and Weyl rescaling. If they do not exist, the gauge is possible globally.

The dimension of the space of zero modes of $P_1^\dagger$ is finite, and thus one finds that the conformally inequivalent world sheets of a given topology are described by a finite number of parameters called "moduli", which we denote as t_i. The number of moduli depend on the genus of the surface: there are none for S^2, there are two real moduli for the torus T^2 and $6g - 6$ real moduli for Riemann surfaces with $g > 2$.

In a symbolic notation one can write the gauge fixing equation for the metric as

$$h_{ab} = e^{2\rho} e^{\xi} \prod_i t_i \; \hat{h}_{ab} \tag{127}$$

where ρ generates the Weyl transformation, ξ the diffeomorphism, t_i represent the moduli and $\hat{h}_{ab}$ is a fiducial metric on $\mathcal{M}$. Note that this expression should be modified for $g = 0$ (where there are no moduli parameters) and $g = 1$ (where there are two real parameters).

We can now change variables in the path integral. A change of variables introduces a Jacobian, which in this case is the Fadeev-Popov determinant, det $(P_1^\dagger P_1)$, so finally the measure can be written as

$$\mathcal{D}h = \det(P_1^\dagger P_1) \; \mathcal{D}\rho \mathcal{D}\xi \prod_{i=1}^{6g-6} dt_i. \tag{128}$$

so finally we find

$$Z = V_{st} \int \mathcal{D}\rho\mathcal{D}\xi V_{diff} \times V_{Weyl} \prod_{i=1}^{6g-6} dt_i \det(P_1^\dagger P_1) \left(\frac{\det'\Delta_h}{\int \sqrt{h} d^2\sigma} \right)^{-D/2} \tag{129}$$

We now have to determine the dependence of the determinants on the Weyl factor. This can be done, for example, using the heat kernel. Their behavior under Weyl rescaling is given by

$$\begin{aligned} \left[\det P_1^\dagger P_1\right]^{1/2} &\propto \left[\det \hat{P}_1^\dagger \hat{P}_1\right]^{1/2} e^{-26 S_L(\rho)} \\ \left[\frac{\det'\Delta_h}{\int_{\mathcal{M}} d^2\sigma\sqrt{h}}\right]^{1/2} &\propto \left[\frac{\det'\Delta_{\hat{h}}}{\int_{\mathcal{M}} d^2\sigma\sqrt{\hat{h}}}\right]^{1/2} e^{-S_L(\rho)} \end{aligned} \tag{130}$$

where

$$S_L = \frac{1}{12\pi} \int d^2z \sqrt{\hat{h}} \left(R(\hat{h})\rho + \frac{1}{2}\hat{h}^{ab}\partial_a\rho\partial_b\rho + \mu_0^2(e^{2\rho} - 1) \right). \tag{131}$$

was called by Polyakov the Liouville action.

Finally it is now possible to define the measure over the diffeomorphisms as

$$||d\xi_a||^2 = \int_{\mathcal{M}} d^2z \sqrt{h} h^{ab} d\xi_a d\xi_b. \tag{132}$$

Note that it is Weyl invariant. This allows to factorize the volume of the diffeomorphism group (actually the volume of the diffeomorphisms orthogonal to those generated by the conformal Killing vectors) and the partition function results

$$Z = \sum_g e^{-\lambda\chi} \int \frac{\mathcal{D}\rho}{V_{CKV} \times V_{Weyl}} \int \prod_{i=1}^{6g-6} dt_i (\det'\hat{P}_1^\dagger \hat{P}_1)^{1/2} \left(\frac{\det'\Delta_{\hat{h}}}{\int \sqrt{\hat{h}} d^2 z)} \right)^{-D/2} e^{(D-26)S_L} \tag{133}$$

where the prime in the first determinant accounts for the exclusion of the zero modes of P_1. A diffeomorphism generated by a conformal Killing vector corresponds to a change in the conformal factor and we should count each deformation of the metric only once.

Therefore the partition function can be finally written as

$$Z = \sum_g e^{\lambda\chi} \frac{1}{V_{CKV}} \int \prod_{i=1}^{6g-6} dt_i (\det\hat{P}_1^\dagger \hat{P}_1)^{1/2} \left(\frac{\det'\Delta_{\hat{h}}}{\int (\hat{h})^{1/2} d^2\sigma)} \right)^{-D/2} \int \frac{\mathcal{D}\rho}{V_{Weyl}} e^{(D-26)S_L} \tag{134}$$

This expresion clearly shows the difference between *critical strings* (which in this case means $D = 26$) in which the Liouville action appears with zero coefficient, and $non - critical\ strings$, for all other values of D. If the Liouville field decouples then

Weyl invariance is maintained at the quantum level. In this case the integral over ρ cancels the volume of the Weyl group in the denominator One of the major difficulties in understanding Liouville theory comes from the fact that the path integral measure $\mathcal{D}\rho$ is $||\delta\rho||^2 = \int e^{2\rho}\delta\rho^2$ is not translationally invariant. There have been some suggestions as to how to quantize this theory but they lead to a central charge of at least 25 for the Liouville field and so the only matter that can couple to it has $c < 1$, which is not enough for a string interpretation.

2.5 The partition function

For $g = 0, 1$ the expresion (134) has to be modified. For $g = 0$ there are no moduli parameters and the determinants are constants. There is an infinite factor in the denominator, V_{CKV}. The Riemann Roch theorem states that the dimension of the space of zero modes of P_1 is 6 on the sphere. So the infinite factor can only be canceled if the vacuum expectation value of the vertex operators involves a similar infinite factor. This can indeed happen because the vertex operators are integrals over the sphere of local functions, but we need at least three vertex operators. So the general formula (134) for the tree level scattering amplitude implies that the 0-, 1- and 2-point function vanish. Physically a zero partition function indicates that the spacetime cosmological constant vanishes at tree level. In general relativity it is possible to add the term $\Lambda G^{\mu\nu}$ to the matter energy momentum tensor, where Λ is the cosmological constant. In the classical theory a nonzero value for Λ is equivalent to an extra energy density that is not generated by the matter in the universe. Unfortunatelly even if it is set equal to zero in the classical theory the change of the vacuum energy due to quantum fluctuations generically results in a huge cosmological constant in conventional field theories. In string theory, the vanishing of the cosmological constant at tree-level is an indication that Minkowski spacetime is a solution of the theory. If one tried to quantize a free string in a spacetime with nonzero cosmological constant (for instance de Sitter or anti de Sitter) one runs into a failure of conformal invariance.

Let us see what happens at one loop. The two torii in figure 10 are clearly not conformally equivalent.

For $g = 1$ there are two real moduli. In fact a torus may be defined by specifying two linearly independent lattice vectors on the plane and identifying points that differ by an integer combination of these vectors. Consider the complex z-plane and pick two complex numbers λ_1 and λ_2 as shown in Figure 11. The torus is defined by making the following identifications on the complex plane

$$z \approx z + n\lambda_1 + m\lambda_2, \quad n, m \in Z, \quad \lambda_1, \lambda_2 \in C. \tag{135}$$

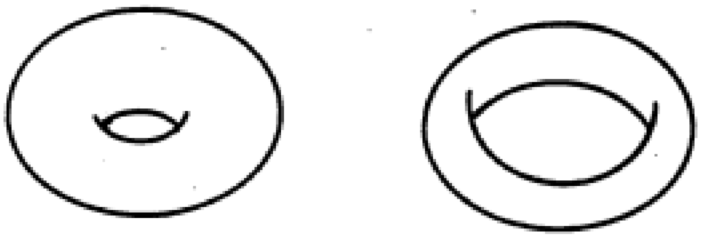

Figure 10: Figure 10.

We want to ask whether tori defined using different values of λ_1 and λ_2 are equivalent under diffeomorphisms plus Weyl rescalings. It is clear that only the ratio $\tau = \lambda_2/\lambda_1$ can be a diffeomorphism plus Weyl invariant. Indeed, a complex rescaling of z, $z' = \alpha\, z$ changes the metric of the torus only by $|\alpha|^2$, which can be absorbed in a conformal rescaling. Thus the transformation rescales λ_1 and λ_2, while leaving fixed the ratio τ. It is thus only τ that is conformally invariant. We can therefore set $\lambda_1 = 1$ and also, because of the freedom to interchange λ_1 and λ_2 we may restrict to Im $\tau > 0$. The tori are thus characterized by points τ in the upper half plane as shown in Figure 12, where opposite sides of the parallelogram are identified:

$$z \approx z + n + m\tau, \quad n, m \in Z \tag{136}$$

It is not quite true that τ is a conformal invariant that cannot be changed by diffeomorphisms and Weyl rescalings. Take the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, b, c, d integers and $ad - bc = 1$ (the area preserving condition). These integer valued matrices of determinant one form a group called the modular group $SL(2, Z)$.

Suppose that we transform λ_1 and λ_2 by an element of $SL(2, Z)$ to λ_1', λ_2'. Then the torus defined by

$$z \approx z + n'\lambda_1' + m'\lambda_2' \tag{137}$$

is exactly the same as that defined by (135); indeed, (135) can be converted into (137) by transforming n and m by the inverse matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

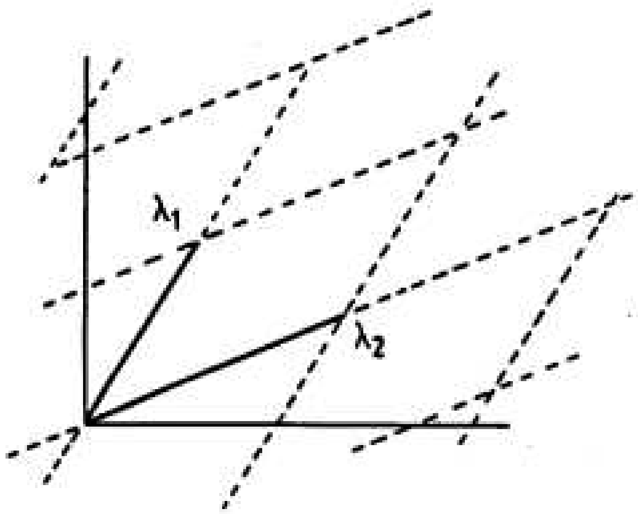

Figure 11: Figure 11.

The fact that (135) and (137) define equivalent tori means that the conformal structure of a torus is invariant under a certain action of the modular group on τ; this action is

$$\tau \rightarrow \frac{(a\tau + b)}{(c\tau + d)}. \tag{138}$$

Actually, the sign of the parameters a, b, c, d may be simultaneously changed without affecting the transformation. Therefore the symmetry of interest is the modular group $SL(2,Z)/Z_2$ or $PSL(2,Z)$. The complex number τ subject to the equivalence relation above is the only feature of the metric of a torus that cannot be absorbed in a diffeomorphism plus Weyl transformation. One often uses the following transformations as the generators of the modular group

$$\begin{aligned} T: &\quad \tau \rightarrow \tau + 1 \\ S: &\quad \tau \rightarrow -\frac{1}{\tau} \end{aligned} \tag{139}$$

Any point in the upper half plane is, via a $PSL(2,Z)$, transformation related to a point in the so called fundamental region $\mathcal{F}$ of the modular group. It is given by $\mathcal{F} = \{-\frac{1}{2} \leq Re\ \tau \leq 1/2, |\tau|^2 \geq 1\}$. $\mathcal{F}$ is shown in Figure 13.

It is the moduli space of the torus and points in $\mathcal{F}$ that describe inequivalent tori. Any non-trivial modular transformation takes τ out of the fundamental region.

We can now compute the partition fuction by explicitly evaluating the determinants in (134). However we will take a different route and work in the operator formalism.

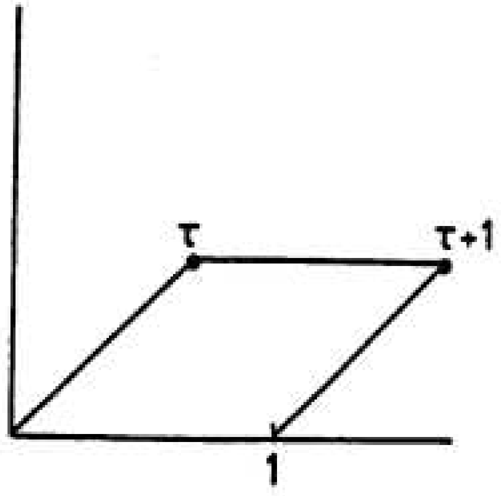

Figure 12: Figure 12.

We can think of the torus with modulus τ as formed by taking a field theory on a circle, evolving for Euclidean time $2\pi\tau_2$, translating in σ by $2\pi\tau_1$, and then identifying the ends. The partition function is then obtained in operator language as a trace,

$$\begin{aligned} Z(\tau) &= Tr\left[exp(2\pi i\tau_1 P - 2\pi\tau_2 H)\right] \\ &= (q\bar{q})^{-D/24} Tr\left(q^{L_0}\bar{q}^{\bar{L}_0}\right). \end{aligned} \tag{140}$$

Here the momentum $P = L_0 - \bar{L}_0$ generates translations of σ and the Hamiltonian $H = L_0 + \bar{L}_0 - \frac{1}{24}(D + \bar{D})$ generates translations in time. The constant term has to be added on the cylinder because T_{ab} is not a tensor, thus it picks up this term when changing coordinates from the plane to the cylinder. We have defined $q = \exp(2i\pi\tau)$.

The trace breaks up into a sum over occupation numbers N_m and $\bar{N}_m$ and an integral over momentum k^μ. The Virasoro generators similarly break up into a sum, and $Z(\tau)$ becomes

$$Z(\tau) = V(q\bar{q})^{-D/24}\int\frac{d^Dp}{(2\pi)^D}exp(-\pi\tau_2\alpha' p^2)\prod_{\mu,n}\sum_{N_m,\bar{N}_m}^{\infty} q^{nN_m}\bar{q}^{n\bar{N}_m} \tag{141}$$

The factor of spacetime volume V comes from the continuum normalization of the momentum, $\sum_p$ becoming $V(2\pi)^{-D}\int d^Dp$. The sums are geometric,

$$\sum_{N=0}^{\infty} q^{nN} = (1 - q^n)^{-1}, \tag{142}$$

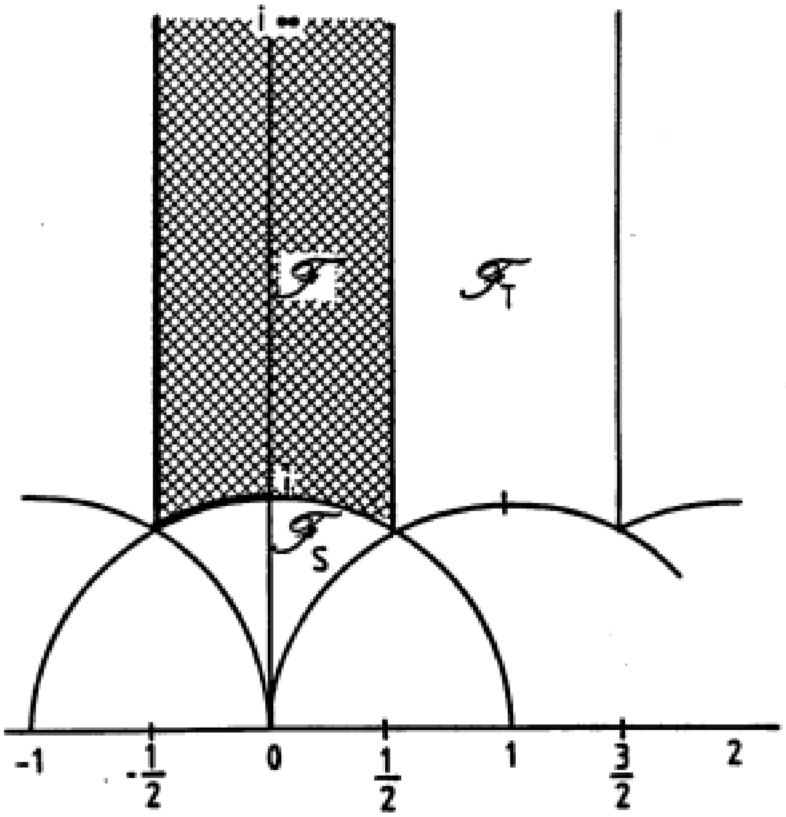

Figure 13: Figure 13.

so we finally obtain

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{8\pi^2(\tau_2)^2} \frac{1}{(4\pi^2\tau_2)^{12}} |\eta(\tau)|^{-48}, \tag{143}$$

where η is the Dedekind function,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{144}$$

We could have followed an alternative procedure and explicitly computed the determinants in (134). They are

$$(\det{}' \hat{P}_1^\dagger \hat{P}_1)^{1/2} = \frac{1}{2} \det{}' \Delta_{\hat{h}} = \frac{1}{2} \tau_2^2 |\eta(\tau)|^4, \tag{145}$$

and of course lead to the same result. The extra factor τ_2^{-2} in (143) comes from V_{CKV} and it makes the measure modular invariant. In fact, under modular transformations

$$d^2\tau \to |c\tau + d|^{-4} d^2\tau$$

$$\text{Im } \tau \to |c\tau + d|^{-2}\text{Im } \tau \tag{146}$$

The modular invariance of the remaining piece in the integral can be checked using the modular transformation properties of η, namely

$$\begin{aligned} \eta(\tau+1) &= e^{i\pi/12}\eta(\tau) \\ \eta(-\frac{1}{\tau}) &= \sqrt{-i\tau}\eta(\tau) \end{aligned} \tag{147}$$

The one loop partition function (143) is divergent. The scattering amplitudes including external states are also divergent for all values of momenta, due to the factor $|\eta(\tau)|^{-48}$ which contributes a factor exponential in τ_2 as $\tau_2 \to \infty$. Ultimately this is connected with the presence of the tachyon. Other manifestations of the instability of Minkowski spacetime are the fact that the one loop dilaton tadpole is nonvanishing, indicating that the Minkowski spacetime is not a solution to the string equations of motion to this order.

It is instructive to compare this result with the analogue calculation of the quantum corrections to Λ in ordinary field theory with point particles. The expression for the cosmological constant is given by the sum of the vacuum (or zero-point) energy of each species of particle. These quantum corrections to Λ can be determined from the term in the quantum effective action of the form $\int d^Dx\sqrt{G}\Lambda$. To lowest order in perturbation theory this depends only on the kinetic terms in the field theory action. Integrating over the field degrees of freedom in the partition function gives the expression for Λ arising from one particular species of particle with mass m,

$$\Lambda = \pm\frac{1}{2}\ln\,\det(p^2+m^2) = \pm\frac{1}{2}\int\frac{d^Dp}{(2\pi)^D}\ln(p^2+m^2), \tag{148}$$

(+ for a boson and $-$ for a fermion). This is obviously divergent in the absence of a regulator. In order to discuss the string theory calculation it is useful to rewrite Λ in the form

$$\Lambda = \pm\frac{1}{2}lim_{s\to 0}\frac{\partial}{\partial s}\left[\frac{1}{\Gamma(-s)}\int_0^\infty t^{-s-1}\int\frac{d^Dp}{(2\pi)^D}e^{-t(p^2+m^2)}dt\right], \tag{149}$$

where the identity $\ln\, a = \lim_{s\to 0}(\partial a^s/\partial s)$ has been used together with an integral representation for a^s. If we ignore the divergence at $t=0$ for the moment, this expression can be written as

$$\Lambda = \mp\frac{1}{2}\int_0^\infty\frac{dt}{t}\int\frac{d^Dp}{(2\pi)^D}e^{-t(p^2+m^2)}, \tag{150}$$

(where $-$ refers to bosons and + to fermions). The fact that the integral diverges at $t=0$ is a symptom of ultraviolet divergences that need regularizing in ordinary field theories.

In order to compare with the string theory calculation let us explicitly write the τ integral of expression (141)

$$Z = -\frac{1}{2}\int \frac{d^{26}p}{(2\pi)^{26}} \int_{\mathcal{F}} d^2\tau \frac{1}{(Im\tau)^2} e^{-\pi\alpha' p^2 Im\tau} tr\left(q^N \bar{q}^{\bar{N}}\right) \tag{151}$$

If we took the integration region to be the infinite strip consisting of $0 < \text{Im}\ \tau < \infty$, $-1/2 \leq \text{Re}\tau \leq 1/2$ then this would correspond to (149). But if we restrict to the fundamental region we cut off the ultraviolet contributions to the vacuum energy of the component string states. The cosmological constant is still infrared divergent in the bosonic string theory due to the tachyon contribution, but the presence of the ultraviolet cut off is a significant departure from the point particle theories.

The results for the torus can be extended to the other surfaces of Euler number zero. For example the vacuum amplitude from the cylinder in the oriented theory is

$$\begin{aligned} Z_{C_2} &= \int_0^\infty \frac{dt}{2t} Tr'_o[exp(-2\pi t L_0)] \\ = iV_D \int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-D/2} &\sum exp[-2\pi t(h_i - 1)] \\ iV_{26} n^2 \int_0^\infty & \frac{dt}{2t}(8\pi^2\alpha' t)^{-13}\eta(it)^{-24} \end{aligned} \tag{152}$$

In the last line the trace has been carried out for the case of 26 flat dimensions, with n Chan-Paton degrees of freedom.

We will work in $D = 26$, *i.e.* with critical strings.

3 Interactions

Let us now consider tree-level scattering amplitudes. We have seen that correlation functions involving one and two vertex operators vanish on the sphere due to the infinite factor V_{CKV} in the path integral. This indicates that flat spacetime is a tree level solution of the string equations of motion and that there are no tree level mass and wave function corrections. Indeed in quantum field theory we describe a vacuum state by choosing vacuum expectation values of fields $\Phi_0 =< \Phi >$ and writing $\Phi = \Phi_0 + \phi$, where ϕ is the quantum fluctuation. So $< \phi >= 0$ is the statement that the candidate vacuum state around which we are expanding is a solution of the classical field equations.

Assuming that we work in the critical dimension the general formula for tree level scattering amplitudes is

$$A_n^{(0)} = e^{-2\lambda} \frac{1}{V_{CKV}} < V_1(k_1)V_2(k_2)\cdots V_n(k_n) > \tag{153}$$

This can be computed using Wick's theorem and the basic free field two point functions on the sphere

$$< X^\mu(z)X^\nu(w) >= -\eta^{\mu\nu}\ln(z-w) \tag{154}$$

(or the disk if we consider open string theory).

As an example of the results obtained consider the amplitude describing elastic scattering of four closed string ground states. One can deal with the infinite volume in the denominator by fixing three points (it is customary to choose $z_1 = 0, z_2 = 1$ and $z_3 = \infty$) and thus one is left with only one integral in this case. In terms of the usual Mandelstam variables $s = -(k_1+k_2)^2$, $t = -(k_2+k_3)^2$ and $u = -(k_1+k_3)^2$, the result is

$$A(s,t) = e^{-2\lambda}\int d^2z|z|^{2k_1\cdot k_2}|1-z|^{2k_1\cdot k_3} \tag{155}$$

This integral gives the Virasoro Shapiro amplitude

$$< V(k_1)V(k_2)V(k_3)V(k_4) >= e^{-2\lambda}\delta(k)\frac{\Gamma(-1-s/2)\Gamma(-1-t/2)\Gamma(-1-u/2)}{\Gamma(2+s/2)\Gamma(2+t/2)\Gamma(2+u/2)}, \tag{156}$$

This amplitude factorizes at the poles corresponding to the on-shell states in the theory. In particular, the tachyon pole $s \sim -2$ imposes a relation between the three and four point functions which fixes the relative normalization of the 3- and 4-point amplitudes. Moreover this expression exhibits the duality symmetry $s \leftrightarrow t \leftrightarrow u$.

Similarly for the open string four tachyon scattering one obtains

$$A(s,t) = e^{-\lambda}B(-\alpha(s),-\alpha(t)) = e^{-\lambda}\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{157}$$

where the Regge trajectory $\alpha(s)$ is

$$\alpha(s) = 1 + \alpha' s \tag{158}$$

This is the famous Veneziano amplitude.

The most convenient way to summarize the results of these computations in any string theory is to specify the effective field theory action. By definition this effective action is such that if we compute the tree level scattering amplitudes using this action, we should reproduce the S-matrix elements involving the massless states of string theory. In general such an action will have to contain infinite number of terms, but we can organize these terms by examining the number of spacetime derivatives that appear in a given term in the action. Terms with the lowest number of derivatives constitute the low energy effective action, so called because this gives the dominant contribution if we want to evaluate the scattering amplitude when all the external particles have small energy and momenta.

The first indication that string theory is a low energy expansion about a point particle theory arose in a paper by Scherk. Low energy means that the string scale $\alpha' \to 0$. Neveu and Scherk showed that in this limit, the string scattering amplitudes of the massless spin one particles of the open bosonic string can be reproduced by a Yang-Mills field theory. Subsequently it was shown by Yoneya and Scherk and Schwarz that the effective action of the spin two state is, in the zero slope limit, given by the Einstein-Hilbert action of gravitation.

The general procedure to extract the low energy effective field theory from string theory is the following: one first calculates various string scattering amplitudes of massless string states, represented by their vertex operators. Then one writes down a field theory lagrangian which reproduces these amplitudes. This is done in a perturbative fashion. One starts by writing down the effective lagrangian that describes the massless free particles, $\mathcal{L}_{2pt}$. Then one adds $\mathcal{L}_{3pt}$ to reproduce the three point string amplitudes. $\mathcal{L}_{3pt}$ already allows to relate various coupling constants of the effective action to the string coupling constant and the string tension α'. Higher powers of the external momenta will be accompanied by powers of $\sqrt{\alpha'}$, in this way the expansion of the effective action in numbers of derivatives is an expansion in powers of $\sqrt{\alpha'}$. At the next step one considers the four point functions. Unitarity guarantees that the massless poles will be those generated by the tree graphs of $\mathcal{L}_{3pt}$. This allows us to check again the relation between the relevant coupling constants. The remainder is in general due to massive particle exchanges and will be reproduced by $\mathcal{L}_{4pt}$. The contribution to the string four point amplitudes which are due to massive particle exchange can be expanded in powers of the external momenta. Each term in this expansion generates a local four point vertex in $\mathcal{L}_{4pt}$. This procedure can now be carried on to arbitrary order. At each order we will write $\mathcal{L}$ in a form invariant with respect to all the global and local symmetries so that for instance the kinetic energy term for a charged scalar, which is part of $\mathcal{L}_{2pt}$, does already contain the three and four point couplings of the scalars to gauge fields (via gauge invariance) and couplings to arbitrary order of the scalar to gravitons (general coordinate invariance). These couplings do of course have to reproduce the corresponding string amplitudes.

The effective action reproducing the low energy limit of scattering amplitudes involving massless fields in bosonic string theory is

$$S_{leea} = \int d^{10}x\sqrt{-G}\{\frac{e^{-2\phi}}{2\kappa_0^2}\left(-\frac{2(D-26)}{3\alpha'} + \mathbf{R} + 4(\bigtriangledown\phi)^2 - \frac{1}{12}H_{\mu\nu\kappa}H^{\mu\nu\kappa}\right) \\ -e^{-\phi}\frac{g}{4}TrF_{\mu\nu}F^{\mu\nu} + O(\alpha')\} \qquad (159)$$

Note that the closed string kinetic terms are accompanied by $e^{-2\phi}$ from the sphere and the open string kinetic terms by $e^{-\phi}$ from the disk. In order to identify the low energy string computation with Einstein's and Yang-Mills' theories it is necessary to relate the string coupling constant and tension with Planck's and Yang Mill's constants. It is found that $\kappa = 2e^{-\lambda}\alpha'$. Therefore the masses of the states in the string spectrum are of order Planck mass. Thus our main interest is in the scattering involving massless states if we want to make contact with the real world.

It is however only possible to calculate string amplitudes with external on-shell states. This means that the amplitudes do not fix the field theory lagrangian completely. However, appealing to symmetries such as general coordinate invariance, gauge symmetries and supersymmetry (if present) allows us to derive a unique (up to field redefinitions) lagrangian which furnishes a low energy description of a given string model. In this case we are assuming Chan Paton degrees of freedom for the open string. The Yang-Mills vector potential A_μ are written as $n \times n$ matrices, *i.e.* $A_\mu = A_\mu^a \lambda^a$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, and this is the action for a $U(n)$ gauge field.

An alternative way of arriving at an effective action is the so called sigma-model approach. In this lenguage what we do is a calculation to all orders in sigma-model perturbation theory and to lowest order in string perturbation theory.

It is natural to write the action of a string propagating in a curved spacetime replacing $\eta_{\mu\nu}$ in S_P by a general metric $G_{\mu\nu}$,

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \tag{160}$$

The way to include backgrounds of all the massless string states is

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma \sqrt{h} \left[\left(h^{ab} G_{\mu\nu}(X) + i\varepsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' R \Phi(X) \right] + \\ + \frac{1}{2\pi\alpha'} \int_{\partial\mathcal{M}} A_\mu(X) \partial_\tau X^\mu \tag{161}$$

The field $B_{\mu\nu}(X)$ is the antisymmetric tensor, the dilaton involves both Φ and the diagonal part of $G_{\mu\nu}$ and A^μ is the massless open string gauge boson.

The non linear sigma model action is no longer quadratic in X^μ, so the path integral is now an interacting two-dimensional quantum field theory. It is a renormalizable theory: the fields X^μ have dimension zero and so the interactions have dimension two. Nevertheless the couplings, which are the coefficients in the expansion of the metric (and the other fields), are infinite in number (unless some symmetry restricts their form).

We have seen that Weyl invariance is essential to the consistency of string theory. The action (161) will define a consistent string theory only if the two-dimensional quantum field theory is Weyl invariant. Under local Weyl transformations, the terms with $G_{\mu\nu}$ and $B_{\mu\nu}$ in the action are classically invariant while the Φ term is not. Including quantum contributions the general form of the trace of the energy momentum tensor is

$$T^a_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}h^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R - \beta^A_\mu \partial_\tau X^\mu \tag{162}$$

where

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + O(\alpha'^2), \tag{163}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2}\nabla^\rho H_{\rho\mu\nu} + \alpha' \nabla^\rho \Phi H_{\rho\mu\nu} + O(\alpha'^2), \tag{164}$$

$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha' \nabla_\rho \Phi \nabla^\rho \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + O(\alpha'^2). \tag{165}$$

Terms with more derivatives are higher order in the world-sheet α' expansion.

The condition that the world-sheet theory is Weyl invariant is thus

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = \beta^A_\mu = 0. \tag{166}$$

These look like equations of motion. The equation $\beta^G_{\mu\nu} = 0$ resembles Einstein's equation with source terms from the antisymmetric tensor field and the dilaton. $\beta^B_{\mu\nu} = 0$ is the antisymmetric tensor generalization of Maxwell's equation, determining the divergence of the field strength. In fact, these are the equations of motion derived from the low energy effective action (159).

One other qualitative feature of the field equations is readily understood: the field Φ always occurs differentiated, so there is an invariance under X^μ-independent shifts of Φ. This is because such a shift only changes the world-sheet action by a term proportional to the Euler number, and so does not affect local properties like Weyl invariance. In particular the background

$$G_{\mu\nu}(X) = \eta_{\mu\nu}, \quad B_{\mu\nu}(X) = 0, \quad \Phi(X) = \Phi_0 \tag{167}$$

is exactly Weyl invariant for any constant Φ_0. This is just the flat spacetime action with $\lambda = \Phi_0$.

In fact recall the form of the path integral on a surface of genus g,

$$Z = e^{-2\lambda(1-g)} \int \mathcal{D}X \mathcal{D}h e^{-S} \tag{168}$$

and compare with the dilaton coupling in the σ-model action. We see that under

$$\Phi \to \Phi + a \tag{169}$$

(a = constant) the σ-model action changes by

$$S \to S + 2a(1-g) \tag{170}$$

The effect of this in (168) is equivalent to a redefinition of the gravitation coupling $e^{\lambda} \to e^{\lambda - a}$. Thus the value of λ can be absorbed in a shift in the vacuum expectation value of the dilaton. λ is not a fundamental parameter of the theory.

This result can also be checked in the low energy effective action. While (159) appears to contain the gravitation coupling parameter κ_0 as a free parameter, in fact it can be absorbed in a shift in the value of Φ. The value of λ determines the coupling strength between strings, but now we see that this does not mean that there are different string theories with different values of this parameter. Different values of λ correspond not to different theories, but to different backgrounds in a single theory.

The other parameter appearing in the action α' is also not really a free parameter because it is dimensionful. It simply defines the unit of length, and can be absorbed in the definition of X^{μ}.

The emergence of gravity in all string theories is the most striking result in string theory. Its origin can be traced to the existence of the symmetric rank two tensor state (the graviton). This, combined with the result of finiteness of scattering amplitudes, shows that string theory gives a finite quantum theory of gravity.

Changing the fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ in the world-sheet action would seem, from the two dimensional point of view, to give a new theory. From the point of view of the full string theory, one is merely looking at a different background - a different state - in the same theory. This is one of the features that makes string theory attractive. In the Standard Model, and in most attempts to unify the Standard Model within quantum field theory, there are many constants that are not determined by the theory; any values of these constants give a consistent theory. In string theory there are no such free parameters - the coupling constants depend on the state and are determined in principle by the dynamics. Of course, for now this only moves the difficulty elsewhere because we do not understand the dynamics well enough to know the values of the background fields.

It is striking that Einstein's equation turns up in what seems to be a rather out of the way place, as the condition for Weyl invariance of a two dimensional field theory. String theory provides the physical connection between these two ideas. It is also remarkable that, because of the need for Weyl invariance, strings can propagate consistently only in a background that satisfies appropriate field equations.

The condition $D = 26$ came from the R term in T^{a}_{a}. In the non linear sigma model this is generalized to $\beta^{\Phi} = 0$. The leading term in β^{Φ} is proportional to $D - 26$ but there are corrections involving gradients of the fields. Evidently we can have other values of D if the fields are not constant, and in fact exact solutions with $D \neq 26$ are known.

It is often useful to make a field redefinition of the form

$$\tilde{G}_{\mu\nu}(X) = e^{2\Psi(X)} G_{\mu\nu}(X), \tag{171}$$

which is a spacetime Weyl transformation. The Ricci scalar constructed from $\tilde{G}_{\mu\nu}$ is

$$\tilde{\mathbf{R}} = e^{2\Psi}\left[\mathbf{R} - 2(D-1)\nabla^2\Psi - (D-2)(D-1)\partial_\mu\Psi\partial^\mu\Psi\right] \tag{172}$$

Let $\Psi = 2(\Phi_0 - \Phi)/(D-2)$ and define $\tilde{\Phi} = \Phi - \Phi_0$, which has vanishing expectation value. The action becomes

$$S_{eff} = \frac{1}{2\kappa^2}\int d^Dx(-\tilde{G})^{1/2}[-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + \tilde{\mathbf{R}} - \frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + O(\alpha')] \tag{173}$$

where the indices are raised with $\tilde{G}^{\mu\nu}$. So we see the standard gravitational lagrangian density. The constant $\kappa = \kappa_0 e^{\Phi_0}$ is the observed gravitational coupling constant,

$$\kappa = (8\pi G_N)^{1/2} = (2.43 \times 10^18 GeV)^{-1} \tag{174}$$

Commonly $G_{\mu\nu}$ is called the *sigma model metric* or *string metric*, and $\tilde{G}^{\mu\nu}$ the Einstein metric. There is no preferred definition of the metric.

4 Compactification

So far we have described string theory in 26 spacetime dimensions. Since our world is $(3+1)$ dimensional, these are not realistic theories. However one can construct string theories in lower dimensions using the idea of compactification. This idea amounts to take the 26 dimensional spacetime as the product of a (25-d) dimensional compact manifold $\mathcal{M}$ with euclidean signature and a (d+1) dimensional Minkowski space $R^{d,1}$. Then in the limit when the size of the compact manifold is sufficiently small so that the present day experiments cannot resolve this distance, the world will effectively appear to be (d+1) dimensional. Choosing $d = 3$ will give us a (3+1) dimensional theory. Of course we cannot choose any arbitrary manifold $\mathcal{M}$ for this purpose; it must satisfy the equations of motion of the effective field theory that comes out of string theory. There are many known examples of manifolds satisfying these restrictions *e.g.* tori of different dimensions, K3, Calabi Yau manifolds, etc. Instead of going via the effective action one can also directly describe these compactified theories as string theories. For this one needs to modify the string world sheet action in such a way that it describes string propagation in the new manifold $\mathcal{M} \times R^{d,1}$, instead of in flat ten dimensional spacetime. This modifies the world-sheet theory to an interacting non-linear σ-model instead of a free field theory. Consistency of string theory puts restriction on the kind of manifold on which the string can propagate. At the end both approaches yield identical results.

However the notion of compactification should not be taken too literally. With the most general ansatz of constructing four-dimensional string theories, the concept of critical dimension is replaced by the requirement that the central charge of the Virasoro algebra is zero. This can be realized by introducing only four usual string coordinates corresponding to four dimensional Minkowski spacetime and in addition a two-dimensional conformal field theory which has to satisfy the consistency constraints of unitarity, locality, conformal invariance, modular invariance, etc.

The simplest class of compact manifolds, on which we shall focus our attention in the following are tori,*i.e.* product of circles. The effect of this compactification is to periodically identify some of the bosonic fields in the string world-sheet theory- the fields which represent coordinates tangential to the compact circles. One effect of this is that the momentum carried by any string state along any of these circles is quantized in units of $1/R$, where R is the radius of the circle. But this is another novel effect: we now have new states that correspond to strings wrapped around a compact circle. For such states, as we go once around the string, we also go once around the compact circle. These states are known as winding states and play a crucial role in the analysis of duality symmetries.

Realistic compactifications of string theory will be considered in later lectures, but it is interesting to look at the simplest compactification of string theory, in which one or more dimensions are periodically identififed.

4.1 Toroidal compactification of the closed bosonic string

Let us first consider the simplest case of one coordinate compactified on a circle of radius R. It means that for one spatial coordinate, *e.g.* X^{25} we require periodicity such that points on the real axis are identified according to

$$X^{25} \sim X^{25} + 2\pi R\omega, \quad \omega \in Z \tag{175}$$

Thus X^{25} describes the one-dimensional circle S^1. In other words, S^1 is obtained by dividing the real line by the integers times $2\pi R$

$$S^1 = \frac{\Re}{2\pi R\omega} \tag{176}$$

which also defines the equivalence relation (175). Now the coordinate $X^{25}(\sigma, \tau)$, $0 \leq \sigma \leq 2\pi$, maps the closed string onto the spatial circle $0 \leq X^{25} \leq 2\pi R\omega$. Therefore we have to reformulate the periodicity condition a closed string has to obey in the following way:

$$X^{25}(\sigma+2\pi,\tau)=X^{25}(\sigma,\tau)+2\pi R\omega \tag{177}$$

The second new term describes string states which are only closed on the circle and not on the real axis. These states correspond to so called winding states; they are caracterized by the winding number ω that counts how many times the string wraps around the circle. States with winding numbers $0,1$ and -1 are shown in Figure 14. (a). By the usual process of joining and splitting, a string with $\omega=0$ can become a pair of strings with winding numbers $\omega=\pm1$, as shown in Figure 14. (b). The winding number is always conserved by string interactions. This phenomenon has no counterpart in the theory of point particles. The winding states are topologically stable solitons; the winding number cannot be changed without breaking the string. Such solitons always exist if the internal manifold contains non-contractible loops.

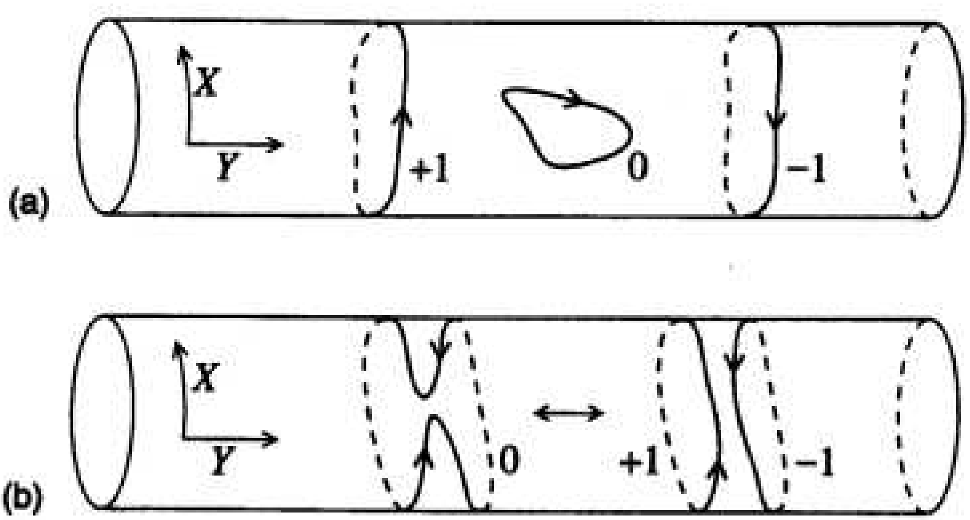

Figure 14: Figure 14.

The general solution to the equations of motion with periodic boundary conditions is $X^\mu(z,\overline{z})=X^\mu(z)+X^\mu(\overline{z})$, with

$$\begin{aligned} X^\mu(z) &= x^\mu+i\sqrt{\frac{\alpha'}{2}}(-\alpha_0^\mu\ ln\ z+\sum_{m\neq0}\frac{\alpha_m^\mu}{m}\ z^{-m}) \\ X^\mu(\overline{z}) &= \tilde{x}^\mu+i\sqrt{\frac{\alpha'}{2}}(-\tilde{\alpha}_0^\mu\ ln\ \overline{z}+\sum_{m\neq0}\frac{\tilde{\alpha}_m^\mu}{m}\ \overline{z}^{-m}) \end{aligned} \tag{178}$$

The spacetime momentum of the string is

$$k^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu) \tag{179}$$

In terms of the world sheet coordinates σ, τ the contribution of the zero modes is

$$X^\mu(\sigma,\tau) = x^\mu + \tilde{x}^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + oscillators \tag{180}$$

Under $\sigma \sim \sigma + 2\pi$, $\delta X^\mu \sim 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)$. For non-compact dimensions, X^μ is periodic so

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}k^\mu \tag{181}$$

For a compact dimension of radius R, say X^{25}, single-valuedness of the wave function $e^{ik^{25}X^{25}}$ restricts the allowed internal momenta to discrete values n/R. Now when $\sigma \sim \sigma + 2\pi$, X^{25} can change as $X^{25} + 2\pi R\omega$, therefore

$$\begin{aligned} \alpha_0^{25} + \tilde{\alpha}_0^{25} &= \sqrt{2\alpha'}\frac{n}{R} \\ \alpha_0^{25} - \tilde{\alpha}_0^{25} &= \sqrt{\frac{2}{\alpha'}}\omega R \end{aligned} \tag{182}$$

and so

$$\begin{aligned} \alpha_0^{25} &= \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} + \frac{\omega R}{\alpha'}\right) \\ \tilde{\alpha}_0^{25} &= \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R} - \frac{\omega R}{\alpha'}\right) \end{aligned} \tag{183}$$

The mass spectrum in 25 dimensions is

$$M^2 = -k_\mu k^\mu = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}\left[\sum_m (\alpha_{-m}\cdot\alpha_m + \tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m) - 2\right] \tag{184}$$

where now $\mu = 0, ..., 24$. Notice that the mass spectrum of the theories at radii R and α'/R are identical after interchanging the winding and Kaluza-Klein modes, $\omega \to n$, which amounts to $\alpha_0^{25} \to \alpha_0^{25}$, $\tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25}$.

Notice also that as we take $R \to \infty$, the decompactification limit, the winding modes become infinitely massive and the compact Kaluza-Klein momentum can take continuous values. This is what one would expect from a non-compact dimension. However the surprise is that as $R \to 0$ the momentum modes become infinitely massive while the winding modes tend to a contimuum (it takes a small amount of energy to wind the string around a very small circle). Thus, as $R \to 0$ the spectrum looks also as the spectrum of the uncompactified theory. This is a stringy effect which has no analogue in the theory of point particles where there is no winding and thus there are no states that become light when $R \to 0$.

The limits $R \to \infty$ and $R \to 0$ are physically equivalent. The spectrum is invariant under $R \leftrightarrow \alpha'/R$, $n \leftrightarrow \omega$, and this equivalence extends to the interactions. This phenomenon is called T-duality, and it evidences that strings behave very differently than point particles. The space of inequivalent theories is the half line $R \geq \alpha'^{1/2}$. We could have taken $0 \leq R \leq \alpha'^{1/2}$ as well. At the non-perturbative level you will see structure at smaller distances.

Let us consider the spectrum of this theory. The mass operator gets contributions from the winding states. We can split the left and right contributions as

$$M_L^2 = \frac{1}{2}\left(\frac{n}{R} + \frac{1}{2}\omega R\right)^2 + N_L - 1,$$
$$M_R^2 = \frac{1}{2}\left(\frac{n}{R} - \frac{1}{2}\omega R\right)^2 + N_R - 1, \tag{185}$$

where $M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu$. The factor $\frac{n^2}{R^2}$ is the contribution from the momenta in the compact dimension and the term $\frac{\omega^2 R^2}{\alpha'^2}$ is the energy required to wrap the string around the circle ω times.

Physical string states have to satisfy the constraint

$$M_L^2 = M_R^2, \quad N_R - N_L = \omega n \tag{186}$$

Let us examine the spectrum of the effectively 25-dimensional string theory. First consider states with no winding number and internal momentum excitations.

i) The lowest energy state is again the scalar tachyon with $M^2 = -2$ (we supress the spacetime momentum)

$$|tachyon >= |0> . \tag{187}$$

ii) At the massless level with $N_L = N_R = 1$ there are now the 25 dimensional graviton, antisymmetric tensor and dilaton.

$$|G^{\mu\nu} >= \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0>, \quad \mu,\nu = 2, \cdots 24 \tag{188}$$

with the oscillators in the 23 uncompactified spacetime directions (we are working in the light-cone gauge).

iii) In addition to these states which were already present in the uncompactified theory there are also new states which arise from the compactification. We can replace one spacetime oscillator by an internal oscillator to get two vector states:

$$|V_1^\mu >= \alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1}|0>,$$
$$|V_2^\mu >= \alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1}|0> . \tag{189}$$

These massless vectors originate from the Kaluza Klein compactification of the bosonic string on the circle - they are just part of the original 26-dimensional graviton and antisymmetric tensor field. They give rise to a $U(1)_L \times U(1)_R$ gauge symmetry which corresponds to the left and right isometries of the circle. Of course, the appearance of these two gauge bosons is expected in any field theoretical compactification.

iv) Finally, acting with two internal oscillators on the vacuum we obtain a massless scalar field which is also a compactified degree of freedom of the 26-dimensional metric; its vacuum expectation value corresponds to the radius R of the circle

$$|\phi> = \alpha_{-1}^{25} \tilde{\alpha}_{-1}^{25} |0> . \tag{190}$$

Let us now turn to the more interesting case, namely states with nontrivial internal momentum and winding number. Therefore we act now with the oscillators on the soliton vacua $|n, \omega>$. We will concentrate on the first winding sector $n = \pm\omega = \pm 1$.

Choosing $\omega = n = \pm 1$ we derive from eq. (185) that

$$\begin{aligned} M_L^2 &= \frac{1}{2R^2} + \frac{1}{8}R^2 + N_L - \frac{1}{2} \\ M_R^2 &= \frac{1}{2R^2} + \frac{1}{8}R^2 + N_R - \frac{3}{2} \\ M^2 &= \frac{1}{R^2} + \frac{1}{4}R^2 + N_L + N_R - 2 \end{aligned} \tag{191}$$

The level matching constraint is satisfied if $N_L = 0, N_R = 1$. Thus we have two vector states of the form

$$|V_a^\mu> = \alpha_{-1}^\mu | \pm 1, \pm 1>, \quad a = 1, 2, \quad \mu = 2, \cdots, 24 \tag{192}$$

and also two scalars

$$|\phi_a> = \alpha_{-1}^{25} | \pm 1, \pm 1>, \tag{193}$$

with mass which depends on the radius of the circle

$$M^2(R) = \frac{1}{R^2} + \frac{1}{4}R^2 - 1. \tag{194}$$

(we have taken $\alpha' = 1$). Analogously we can set $\omega = -n = \pm 1$. Then the matching condition is satisfied if $N_L = 1, N_R = 0$ and we obtain again two vectors and two scalars

$$\begin{aligned} |V_a'^\mu> &= \tilde{\alpha}_{-1}^\mu | \pm 1, \mp 1>, \quad a = 1, 2, \quad \mu = 2, \cdots, 24 \\ |\phi_a'> &= \tilde{\alpha}_{-1}^{25} | \pm 1, \mp 1>, \end{aligned} \tag{195}$$

With mass also given by (194). It is easy to see that $M^2(R) \geq 0$ with equality holding for $R = \sqrt{2}$. This means that for this particular radius, which is determined by the string tension, we get extra massless states, of which the massless vectors are of particular interest. This phenomenon is of utmost importance in the theory of compactified bosonic string. The four additional massless vectors form, together with the massless vectors of (189), the adjoint representation of $SU(2)_L \times SU(2)_R$. The oscillator excitations with zero winding number correspond to the $U(1)_L \times U(1)_R$ Cartan subalgebra generators of $SU(2)_L \times SU(2)_R$.

We have seen that for a special value of the radius of the compact circle one gets an enhancement of the gauge symmetry due to the soliton states. This can never occur in any point particle compactification. Furthermore, for $R = \sqrt{2}$ we get also eight additional massless scalars

$$\alpha_{-1}^{25}|\pm 1, \pm 1 >, \tilde{\alpha}_{-1}^{25}|\pm 1, \mp 1 >, |\pm 2, 0 >, \alpha_{-1}^{25}|0, \pm 2 > \tag{196}$$

which together with $\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|0,0>$ form the (3,3) representation of $SU(2)_L \times SU(2)_R$. However for arbitrary values of the radius, both the four non-commuting gauge bosons of $SU(2)_L \times SU(2)_R$ and the four scalars with internal oscillator excitations are massive - the gauge symmetry is broken to $U(1)_L \times U(1)_R$. Therefore this phenomenon can be interpreted as a stringy Higgs effect. For arbitrary radii these four massive scalars build the longitudinal components of the four massive vector particles. Two of the remaining scalars in eq. (196) become massive and the other two become tachyons.

The $U(1)_L \times U(1)_R$ gauge bosons on the other hand stay massless for all values of R. This is also the case for the single scalar of (190). In a low energy effective field theory this neutral scalar will have a completely flat potential which corresponds to the freedom of choosing the radius of the circle as a free parameter. In summary, the spectrum of the bosonic string compactified on S^1 is characterized by a single parameter, also called modulus, namely the radius of the circle which is the vacuum expectation value of the scalar field (190).

Let us see how the $SU(2)_L \times SU(2)_R$ gauge symmetry arises. The Kaluza-Klein gauge symmetry originates in the X^{25} translations, so the corresponding charge is the compact momentum k^{25}.

For the gauge symmetry of the antisymmetric tensor, let us consider the vertex operator at zero momentum, which measures the charge of any state to which it couples,

$$\partial X^\mu \bar{\partial} X^{25} - \partial X^{25} \bar{\partial} X^\mu = \bar{\partial}(X^{25}\partial X^\mu) - \partial(X^{25}\bar{\partial} X^\mu) \tag{197}$$

It is a total derivative, but it does not integrate to zero on a winding state because X^{25} is not singlevalued. Therefore the charge $B_{\mu 25}$ is the winding number.

Consider now the vertex operators of the four extra gauge bosons with nonzero winding and compact momentum, namely

$$n = \omega = \pm 1, N = 0, \bar{N} = 1 \rightarrow \bar{\partial} X^\mu e^{ik\cdot X} e^{\pm 2iX_L^{25}/\sqrt{2}} \tag{198}$$

$$n = -\omega = \pm 1, N = 1, \bar{N} = 0 \rightarrow \partial X^\mu e^{ik\cdot X} e^{\pm 2iX_R^{25}/\sqrt{2}} \tag{199}$$

These states have internal momentum and winding, thus they carry Kaluza-Klein and winding charges. The only consistent theory of charged massless vectors is non Abelian gauge theory, so the new gauge bosons have to combine with the previous $G_{\mu 25}, B_{\mu 25}$ to form a non Abelian theory.

Working in the base $\partial X^\mu \bar{\partial} X^{25}$, $\bar{\partial} X^\mu \partial X^{25}$ which corresponds to the original gauge bosons ($\omega = n = 0$), it is easy to see that they couple to k_R^{25} and k_L^{25} respectively. The vertices (198) have zero k_R^{25} charge, and $\pm\sqrt{2}\, k_L^{25}$ charge, whereas the vertices (199) have charges $\pm\sqrt{2}$ and 0 respectively. This identifies the gauge group as $SU(2)_L \times SU(2)_R$, with the three generators involving ∂X^μ forming one $SU(2)$ and the three involving $\bar{\partial} X^\mu$ forming the other. To exhibit the $SU(2) \times SU(2)$ symmetry we can define the three left currents

$$\begin{aligned} J^1(z) &= cos(X_L^{25}/\sqrt{2}) \\ J^2(z) &= sin(X_L^{25}/\sqrt{2}) \\ J^3(z) &= i\frac{\partial X_L^{25}}{\sqrt{2}} \end{aligned} \tag{200}$$

and similarly for the right currents $J(\bar{z})$. These holomorphic (antiholomorphic) currents can be Laurent expanded

$$J^i(z) = \sum_{m=-\infty}^{\infty} \frac{J_m^i}{z^{m+1}} \tag{201}$$

and the Laurent coefficients satisfy an infinite dimensional current algebra

$$[J_m^i, J_n^j] = \frac{m}{2}\delta_{m+n}\delta^{ij} + i\epsilon^{ijk} J_{m+n}^k \tag{202}$$

This is called current algebra or Kac-Moody algebra.

In a theory of point particles only the $U(1) \times U(1)$ symmetry is evident at any radius.

This mechanism can be generalized to T^D, and this will be discussed by Narain next week.

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